SOLVING QUARTICS
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1. Method

First note that by replacing x by $x - d/4$ we can eliminate the cubic term from $x^4 + dx^3 + ax^2 + bx + c$. Thus, we assume that there is no cubic term.

We propose to factor

$$x^4 + ax^2 + bx + c = (x^2 + \alpha_1 x + \beta_1)(x^2 + \alpha_2 x + \beta_2).$$

This results in the equations

$$\begin{aligned} \alpha_1 + \alpha_2 &= 0 \\ \beta_1 + \alpha_1\alpha_2 + \beta_2 &= a \\ \alpha_1\beta_2 + \alpha_2\beta_1 &= b \\ \beta_1\beta_2 &= c \end{aligned}$$

The first equation allows us to replace α_2 by $-\alpha_1$, resulting in the three equations

$$\begin{aligned} \beta_1 + \beta_2 &= a + \alpha_1^2 \\ -\beta_1 + \beta_2 &= b/\alpha_1 \\ \beta_1\beta_2 = c \end{aligned}$$

The first two equations here give us that the $2\beta_i$ are the sum and difference of $a + \alpha_1^2$ and b/α_1, which we then insert into the third to get

$$(a + \alpha_1^2 + \frac{b}{\alpha_1})(a + \alpha_1^2 - \frac{b}{\alpha_1}) = 4\beta_1\beta_2 = 4c$$

or

$$(a + \alpha_1^2)^2 - \frac{b^2}{\alpha_1^2} = 4c$$

Letting $A = \alpha_1^2$, this gives the cubic

$$A^3 + 2aA^2 + (a^2 - 4c)A - b^2 = 0 \tag{1}$$

which we solve for A. We then have the solution:

$$\begin{aligned} \alpha_1 &= \sqrt{A} \\ \alpha_2 &= -\sqrt{A} \\ \beta_1 &= (a + A - \frac{b}{\alpha_1})/2 \\ \beta_2 &= (a + A + \frac{b}{\alpha_1})/2 \end{aligned}$$

Now we solve the two quadratics, and we're done.

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2. An example

Here is an example. Consider

$$x^4 - 15x^2 + 10x + 24.$$

The cubic we must solve is

$$A^3 - 30A^2 + 129A - 100,$$

which has roots $A = 25$, $A = 4$ and $A = 1$.
If we use the root $A = 25$ we get the quadratics

- $x^2 + 5x + 4 = (x + 1)(x + 4)$ and
- $x^2 - 5x + 6 = (x - 2)(x - 3)$.

If we use the root $A = 4$ we get the quadratics

- $x^2 + 2x - 8 = (x - 2)(x + 4)$ and
- $x^2 - 2x - 3 = (x + 1)(x - 3)$.

If we use the root $A = 1$ we get the quadratics

- $x^2 + x - 12 = (x - 3)(x + 4)$ and
- $x^2 - x - 4 = (x - 2)(x + 1)$.

In each case, we find that $x^4 - 15x^2 + 10x + 24 = (x + 1)(x - 2)(x - 3)(x + 4)$.

3. A better example

Consider the twelfth cyclotomic polynomial:

$$x^{12} - 1 = (x - 1)(x + 1)(x^2 + 1)(x^2 + x + 1)(x^2 - x + 1)(x^4 - x^2 + 1).$$

The roots are 1, -1, $\pm i$, the primitive third roots of 1, the primitive sixth roots of 1, and the primitve twelfth roots of 1, respectively. Thus we focus on the last factor, which is $\Phi_{12}(x)$.

The cubic we must solve is

$$A^3 - 2A^2 - 3A = A(A + 1)(A - 3)$$

so we can take $A = 0$, $A = -1$, or $A = 3$. The first doesn't fit our general scheme very well, since it requires division by 0, so we will return to it in a moment.

If $A = -1$, we have

$$x^4 - x^2 + 1 = (x^2 + ix - 1)(x^2 - ix - 1) = (x - \frac{-i + \sqrt{3}}{2})(x - \frac{-i - \sqrt{3}}{2})(x - \frac{i + \sqrt{3}}{2})(x - \frac{i - \sqrt{3}}{2})$$

If $A = 3$, we have

$$x^4 - x^2 + 1 = (x^2 + \sqrt{3}x + 1)(x^2 - \sqrt{3}x + 1) = (x - \frac{-i - \sqrt{3}}{2})(x - \frac{i - \sqrt{3}}{2})(x - \frac{-i + \sqrt{3}}{2})(x - \frac{i + \sqrt{3}}{2})$$

If we let ζ be the quotient ω/i of a cube root of one and a fourth root of 1, we see that ζ is the primitive twelfth root of one

$$\zeta = \exp(2\pi i/12) = (\sqrt{3} + i)/2.$$

The other three primitive twelfth roots of one are then ζ^5, ζ^7, and ζ^{11}. Then choosing $A = -1$ corresponds to have grouped the primitive twelfth roots of one into the sets

$$\{\zeta^7, \zeta^{11}\} \cup \{\zeta, \zeta^5\},$$

while choosing $A = 3$ corresponds to the grouping

$$\{\zeta^1, \zeta^{11}\} \cup \{\zeta^5, \zeta^7\}.$$

The remaining division,

$$\{\zeta^1, \zeta^7\} \cup \{\zeta^5, \zeta^{11}\} = \{\zeta, -\zeta\} \cup \{\zeta^5, -\zeta\},$$

corresponds to the factorization

$$x^4 - x^2 + 1 = (x^2 - (\omega + 1)))(x^2 - (\omega^2 + 1)))$$

which simply says that a primitive twelfth root of one can be obtained by taking a square root of a primitive sixth root of one. That is, the difficulty we faced, the inability to divide by 0, was mitigated by the unhelpfulness of the answer this division of the roots would have provided.

This appearance of the Klein four-group, the symmetries of the decomposition of a four element set into a pair of two-element sets, neatly fits with the Galois theory of this situation.

4. Comments

The Galois group S_4 of the general quartic surjects onto S_3 with kernel the Klein 4-group, $C_2 \times C_2$:

$$1 \longrightarrow C_2 \times C_2 \longrightarrow S_4 \longrightarrow S_3 \longrightarrow 1$$

This perfectly reflects the steps taken in the solution above:

(1) Solve a cubic.
(2) Solve two quadratics.

$$S_4 \left\{ \begin{array}{c} F = E(\sqrt{\alpha_1^2 - 4\beta_1}, \sqrt{\alpha_2^2 - 4\beta_2}) \\ \Big| \; C_2 \times C_2 \\ E \\ \Big| \; S_3 \\ k \end{array} \right.$$

5. Question

After solving the cubic, we extract a square root. This appears to suggest an extra C_2 between the S_3 and the Klein 4-group, but apparently, this never results in a field extension. What is happening here?

One guess is that $\alpha_1 = \sqrt{A}$ is already in E; another is that $\sqrt{\alpha_2^2 - 4\beta_2}$ is in $E(\sqrt{\alpha_1^2 - 4\beta_1})$.

It could depend upon the Galois group of the cubic (1). Note also that we don't need the splitting field of (1) to proceed: we simply need one root, A, of (1).

Clearly it is time to look at the generic case, $k[\sigma_1, \sigma_2, \sigma_3, \sigma_4] \subset k[x_1, x_2, x_3, x_4]$.

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