THE CONNECTIVE COMPLEX K-THEORY OF AN ELEMENTARY ABELIAN P-GROUP
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Erich Ossa ([1]) computed the complex connective K-theory of elementary abelian groups by exploiting the idempotence of $H^*BZ/(2)$ as a module over the exterior algebra $E[Q_0, Q_1]$. At odd primes, the calculation splits into $(p-1)^2$ pieces, each of which is simply a regrading of the result at $p = 2$.

Here, we use this method to give a precise calculation of $ku_*(BC_p)^r$ focusing on odd primes p. Let $q = 2(p-1)$.

A note on notation: we write ku for the spectrum formerly denoted bu, reserving bu for the connected cover. Thus, ku has zeroth space $BU \times \mathbf{Z}$ while bu has zeroth space BU.

We have splittings

$$ku \;\simeq\; l \;\vee\; \Sigma^2 l \;\vee\; \cdots \;\vee\; \Sigma^{2(p-2)} l$$

where l is the Adams summand with mod p cohomology $H^*l = A/\!\!/E(1)$. Here $E(1) = E[Q_0, Q_1]$, the exterior algebra on Milnor operations Q_0 and Q_1.

There is also a p-local splitting

$$BC_p \;\simeq\; B_1 \;\vee\; B_2 \;\vee\; \cdots \;\vee\; B_{p-1}$$

where B_i is a cell complex with cells in dimensions congruent to $2i-1$ and $2i$ modulo q.

The Adams spectral sequence easily gives

$$l_n B_1 = \begin{cases} \mathbf{Z}/(p^{j+1}) & n = jq+1 \\ 0 & \text{otherwise} \end{cases}$$

It follows that

$$ku_n B_1 = \begin{cases} \mathbf{Z}/(p^{j+1}) & n = jq+2k+1,\ k = 0,\ldots,p-2 \\ 0 & n \text{ even} \end{cases}$$

Using the Thom isomorphism, Lemma 2.2.2 in [2] gives an equivalence $ku \wedge B_{i+1} \;\simeq\; \Sigma^{2i} ku \wedge B_1$, so we shall focus primarily on B_1.

Decomposing the $E(1)$-module $H^*B_1 \otimes H^*B_1$ into indecomposable $E(1)$-modules, we find, using Margolis's theorem on H^*H summands, that

$$l \wedge B_1 \wedge B_1 \;\simeq\; (\Sigma^2 l \wedge B_1) \;\vee\; \Sigma^2 \mathbf{HF}_p[u_q, v_q]$$

where u_q and v_q have degree q. This is Proposition 4.2.3 in [2] with the details of the GEM there made precise, and comes from the characteristic free result, Lemma 4.2.1.

Assembling the summands of ku, we find

$$ku \wedge B_1 \wedge B_1 \;\simeq\; (\Sigma^2 ku \wedge B_1) \;\vee\; \Sigma^2 \mathbf{HF}_p[u_2, v_q]$$

where u_2 has degree 2.

Now summing the middle term in the smash product gives

$$ku \wedge BC_p \wedge B_1 \;\simeq\; (\Sigma^2 ku \wedge BC_p) \;\vee\; \Sigma^2 \mathbf{HF}_p[u_2, v_2]$$

where v_2 also has degree 2.

Finally summing the third term in the smash product gives

$$ku \wedge BC_p \wedge BC_p \;\simeq\; \bigvee_{i=1}^{p-1} ku \wedge BC_p \wedge B_i \;\simeq\; \left(\bigvee_{i=1}^{p-1} ku \wedge \Sigma^{2i} BC_p \right) \;\vee\; \Sigma^2 \mathbf{HF}_p[u_2, v_2, w_2]/(w_2^{p-1}).$$

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This is the result we iterate to describe $ku \wedge (BC_p)^r$ in general. The simplest way to do the bookkeeping is to use the Poincare series of the cohomology before smashing with ku. Since

$$H^*(ku \wedge X) = \mathcal{A} \otimes_{E_1} H^*X$$

this amounts to writing the Poincare series of the $E(1)$-module which becomes the cohomology in question upon tensoring it up from $E(1)$ to $\mathcal{A}$. Of course, this is simply the Poincare series of the cohomology divided by that of $\mathcal{A}/\!\!/E(1)$. Define

- $R = (1-t^q)/(1-t^2)$, the Poincare series of $\mathbf{F}_p[u]/(u^{p-1})$ where u has degree 2. This is the ratio of the Poincare series of H^*ku and H^*l, or of H^*BC_p and H^*B_1.
- $P = t/(1-t)$, the Poincare series of H^*BC_p.
- $E_1 = (1+t)(1+t^{q+1})$, the Poincare series of $E(1)$.

The decomposition of $ku \wedge BC_p \wedge BC_p$ gives the equation

$$RP^2 = t^2R^2P + \frac{t^2R}{(1-t^2)^2}E_1$$

The coefficient of E_1 is the Poincare series for the GEM summand. Since smashing with a GEM produces a GEM, we see by induction that

$$RP^n = c_nRP + f_nE_1$$

for some rational functions c_n and f_n, and that this equation gives the corresponding decomposition of $ku \wedge (BC_p)^{(r)}$. An easy induction provides $c_{n+1} = (t^2R)^n$ and

$$f_{n+1} = \frac{(t^2R)^n + (t^2R)^{n-1}P + (t^2R)^{n-2}P^2 + \cdots + +(t^2R)P^{n-1}}{(1-t^2)^2} = \frac{1}{(1-t^2)^2}\left(\frac{P^{n+1}-(t^2R)^{n+1}}{P-t^2R} - P^n\right).$$

This gives

$$ku \wedge (BC_p)^{(r+1)} \;\simeq\; \left(\bigvee_{i_1,\ldots,i_r=1}^{p-1} \Sigma^{2i} ku \wedge BC_p\right) \;\vee\; HM(f_{r+1})$$

where $i = i_1 + \cdots i_r$ and $M(f)$ is the $\mathbf{F}_p$-vector space with Poincare series f. Reassembling the smash products to get the corresponding result for the Cartesian product, we find

$$ku \wedge (BC_p)^n \;\simeq\; \bigvee_{r=1}^{n} \binom{n}{r} \left\{ \left(\bigvee_{i_1,\ldots,i_{r-1}=1}^{p-1} \Sigma^{2i} ku \wedge BC_p\right) \;\vee\; HM(f_r)\right\} \tag{1}$$

where i and $M(\cdot)$ are as above.

When $n=3$ this specializes to

$$\begin{aligned} ku \wedge (BC_p)^3 \;\simeq\; 3(ku \wedge BC_p) \;\vee\; 3\left\{\left(\bigvee_{i=1}^{p-1} \Sigma^{2i} ku \wedge BC_p\right) \;\vee\; HM(f_2)\right\} \\ \vee\; \left(\bigvee_{i_1,i_2=1}^{p-1} \Sigma^{2(i_1+i_2)} ku \wedge BC_p\right) \;\vee\; HM(f_3) \end{aligned}$$

To compare these results to [1], we first check that the $ku \wedge BC_p$ summands work out correctly. The odd homotopy groups of $ku \wedge BC_p$ have rank $p-1$ after the first few, while the augmentation ideal in the representation ring of $(C_p)^n$ has rank $p^n - 1$. Counting up the summands in (1) we see that we get the correct rank by the identity

$$p^n - 1 = \sum_{r=1}^{n} \binom{n}{r}(p-1)^r.$$

For the GEM summands, our results seem to agree for $n < 3$ and diverge thereafter. If n=p=3, the Poincare series for the GEM according to (1) is

$$\frac{t^2(1+t^2)(3-2t+t^2-t^3+t^4-t^5)}{(1-t^2)^2(1-t)}$$

while [1] gives

$$\frac{t^2(1+t^2)(3/2+t/2+t^2+t^3/2+t^4)}{(1-t^2)^2}.$$

The first discrepancy is in degree 7, where my Poincare series predicts the GEM summand to have rank 9, while Ossa's predicts rank 8. The differences grow thereafter. I have checked that the Poincare series for the cohomology of the two sides of (1) agree out to degree 17 for $p = 3, 5, 7, 11$.

References

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