$\mathcal{A}(2)$-modules and the Adams spectral sequence

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Outline

Tools:

fpmods.py: a sage package for calculating with finitely presented $\mathcal{A}$-modules written by Mike Catanzaro (masters thesis)

ext.1.8.5: the latest version of my C code for calculating minimal resolutions and chain maps for $\mathcal{A}$ and $\mathcal{A}(2)$-modules incoporating Tyler Lawson's dual module code.

Applications:

- Revisit Don Davis' 1975 Bol. Soc. Mat. Mex. paper in which he calculates H^*j and considers its Adams spectral sequence converging to $\pi_* j$.
- Next, find a sequence representing v_2^8. This gives a fairly straightforward calculation of the cohomology of $\mathcal{A}(2)$, simplifying work of Davis and Mahowald from 1982.

Don Davis' conclusion:

THEOREM 1. i) $H^*(bJ)$ *is the* $\mathfrak{a}$*-module with generators* g_0 *and* g_7 (*of degree* 0 *and* 7, *respectively*) *and relations* $Sq^1 g_0$, $Sq^2 g_0$, $Sq^4 g_0$, $Sq^8 g_0 + Sq^1 g_7$, $S^{7s} g_7$, and $(Sq^4 Sq^6 + Sq^7 Sq^3) g_7$.

ii) $\mathrm{Ext}_{\mathfrak{a}}^{s,t}(H^* bJ, Z_2) \approx A^{s,t} \oplus B^{s+2,t+1}$, *where* $A^{s,t} \approx \mathrm{Ext}_{\mathfrak{a}_2}^{s,t}(Z_2, Z_2)$ *without the towers* $h_0^i \omega^{2j+1}$, $i, j \geq 0$, *and* $B^{s,t} \approx \mathrm{Ext}_{\mathfrak{a}_2}^{s,t}(Z_2, Z_2)$ *without* $\omega^i x^{s,t}$ *for all* $x^{s,t}$ *such that* $t - s \leq 3$, *and with infinite towers built upon* $\omega^{2i+1} h_2^2$ *and towers of height four built upon* $\omega^{2i} h_2^2$.

Thus $\mathrm{Ext}_{\mathfrak{a}}^{s,t}(H^*(bJ), Z_2)$ begins as in Table 2. Note that there will be many nonzero differentials in the Adams spectral sequence for $\pi_*(bJ)$. Part (i) im-

The cohomology of j

We start from the fiber sequence given by the Adams conjecture

$$j \longrightarrow ko \xrightarrow{\psi^3 - 1} \Sigma^4 ksp$$

and the known cohomology modules

$$\begin{aligned} H^* ko &= \mathcal{A}//\mathcal{A}(1) \\ H^* ksp &= \mathcal{A}/\mathcal{A}(Sq^1, Sq^2 Sq^3). \end{aligned}$$

It will be useful to also reduce mod 2.

Proposition

$$H^*ko/2 = \mathcal{A}/\mathcal{A}(Sq^2, Sq^{(0,1)}) \qquad \text{and} \qquad H^*ksp/2 = \mathcal{A}/\mathcal{A}(Sq^2 Sq^3).$$

The natural maps $H^*j/2 \longrightarrow H^*j$ *and* $H^*ksp/2 \longrightarrow H^*ksp$ *are the evident quotients by* Sq^1.

Proof.

$$\begin{aligned} H^*ko/2 &= H^*ko \otimes E[Sq^1] \\ &= (\mathcal{A} \otimes_{\mathcal{A}(1)} \mathbf{F}_2) \otimes E[Sq^1] \\ &\cong \mathcal{A} \otimes_{\mathcal{A}(1)} E[Sq^1] \\ &= \mathcal{A} \otimes_{\mathcal{A}(1)} \mathcal{A}(1)/\mathcal{A}(1)(Sq^2, Sq^{(0,1)}) \\ &= \mathcal{A}/\mathcal{A}(Sq^2, Sq^{(0,1)}) \end{aligned}$$

□

Proof.

$$\begin{aligned} H^* ksp/2 &= H^* ksp \otimes E[Sq^1] \\ &= (\mathcal{A} \otimes_{\mathcal{A}(1)} \mathcal{A}(1)/\mathcal{A}(1)(Sq^1, Sq^2Sq^3)) \otimes E[Sq^1] \\ &\cong \mathcal{A} \otimes_{\mathcal{A}(1)} (\mathcal{A}(1)/\mathcal{A}(1)(Sq^1, Sq^2Sq^3) \otimes E[Sq^1]) \\ &= \mathcal{A} \otimes_{\mathcal{A}(1)} \mathcal{A}(1)/\mathcal{A}(1)(Sq^2Sq^3) \\ &= \mathcal{A}/\mathcal{A}(Sq^2Sq^3). \end{aligned}$$

□

Proposition

The maps $H^* ko \longleftarrow H^*\Sigma^4 ksp$ and $H^* ko/2 \longleftarrow H^*\Sigma^4 ksp/2$ induced by $\psi^3 - 1$ each send the generator to $Sq^4 \iota_0$.

Proof.

The homomorphism $(\psi^3 - 1)^* : H^*\Sigma^4 ksp \longrightarrow H^* ko$ is determined by its value in degree 4. This is either Sq^4 or 0:

```
sage: A2 = SteenrodAlgebra(prime=2,profile=(3,2,1))
sage: ko = FP_Module([0],[[Sq(1)],[Sq(2)]],algebra=A2)
sage: ko[4]
[[Sq(4)]]
sage: komod2 = FP_Module([0],[[Sq(2)],[Sq(0,1)]],algebra=A2)
sage: komod2[4]
[[Sq(4)]]
```

It must be Sq^4 since, if it were 0, the Adams spectral sequence would imply that $\pi_3 j$ had order at least 32. The result is also true mod 2 since the map $H^4 ko/2 \longrightarrow H^4 ko$ is an isomorphism. □

Definition

Let C, C_2, K and K_2 be the cokernels and desuspensions of kernels of $(\psi^3 - 1)^*$ and its mod 2 reduction.

$$\begin{array}{ccccccccccc}
0 & \longleftarrow & C & \longleftarrow & H^*ko & \xleftarrow{Sq^4} & H^*\Sigma^4 ksp & \longleftarrow & \Sigma K & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & C_2 & \longleftarrow & H^*ko/2 & \xleftarrow{Sq^4} & H^*\Sigma^4 ksp/2 & \longleftarrow & \Sigma K_2 & \longleftarrow & 0
\end{array}$$

The cokernels of Sq^4 are easy.

Proposition

$$\begin{array}{ccc} C = \mathcal{A}/\mathcal{A}(Sq^1, Sq^2, Sq^4) & \xleftarrow{qo} & H^*ko = \mathcal{A}/\mathcal{A}(Sq^1, Sq^2) \\ \uparrow{\scriptstyle qko} & & \uparrow{\scriptstyle qc} \\ C_2 = \mathcal{A}/\mathcal{A}(Sq^2, Sq^{(0,1)}, Sq^4) & \xleftarrow[qo_2]{} & H^*ko/2 = \mathcal{A}/\mathcal{A}(Sq^2, Sq^{(0,1)}) \end{array}$$

The maps are the evident quotients.

The kernels of Sq^4 are a bit more complicated.

Proposition

$$K = \frac{\Sigma^7 \mathcal{A}}{\mathcal{A}(Sq^1, Sq^7, Sq^{(0,1,1)} + Sq^{(4,2)})}.$$

and

$$K_2 = \frac{\Sigma^7 \mathcal{A}}{\mathcal{A}(Sq^{(4,1)}, Sq^{(0,1,1)} + Sq^{(3,0,1)} + Sq^{(1,3)} + Sq^{(4,2)})}.$$

$$\begin{array}{ccc} H^* ksp = \mathcal{A}/\mathcal{A}(Sq^1, Sq^2 Sq^3) & \xleftarrow{\ io\ } & \Sigma K \\ \Big\uparrow{\scriptstyle qksp} & & \Big\uparrow{\scriptstyle SqK} \\ H^* ksp/2 = \mathcal{A}/\mathcal{A}(Sq^2 Sq^3) & \xleftarrow[io2]{} & \Sigma K_2 \end{array}$$

Here, $io(\Sigma\iota_7) = Sq^4\iota_4$ *and* $io2(\Sigma\iota_7) = (Sq^4 + Sq^{(1,1)})\iota_4$*, while the vertical maps are the evident quotients.*

sage code to compute kernel and cokernel

```
ko = FP_Module([0],[[Sq(1)],[Sq(2)]])
ksp = FP_Module([4],[[Sq(1)],[Sq(2)*Sq(3)]])
sq4 = FP_Hom(ksp, ko, [[Sq(4)]])
ko2 = FP_Module([0],[[Sq(0,1)],[Sq(2)]])
ksp2 = FP_Module([4],[[Sq(2)*Sq(3)]])
sq42 = FP_Hom(ksp2, ko2, [[Sq(4)]])

C,qo = sq4.cokernel()
SK,io = sq4.kernel()
C2,qo2 = sq42.cokernel()
SK2,io2 = sq42.kernel()

qko = FP_Hom(ko2,ko,[[1]])
qksp = FP_Hom(ksp2,ksp,[[1]])
qC = FP_Hom(C2,C,[[1]])

cando,SqK = lift(qksp*io2,io)
```

sage code to print the results

```
print "\nCokernel C degrees: ",C.degs
print "Cokernel C relations: ",C.rels
print "\nCokernel C2 degrees: ",C2.degs
print "Cokernel C2 relations: ",C2.rels

print "\nKernel SK degrees: ",SK.degs
print "Kernel SK relations: ",SK.rels
print "\nKernel SK2 degrees: ",SK2.degs
print "Kernel SK2 relations: ",SK2.rels

print "\nio values: ",io.values
print "io2 values: ",io2.values
print "Lift SqK exists: ",cando
print "SqK values: ",SqK.values
```

sage output

```
sage: load j1.py

Cokernel C degrees:  [0]
Cokernel C relations:  [[Sq(1)], [Sq(2)], [Sq(4)]]

Cokernel C2 degrees:  [0]
Cokernel C2 relations:  [[Sq(0,1)], [Sq(2)], [Sq(4)]]

Kernel SK degrees:  [7]
Kernel SK relations:  [[Sq(1)], [Sq(4,1)], [Sq(0,1,1) + Sq(4,2)]]

Kernel SK2 degrees:  [7]
Kernel SK2 relations:  [[Sq(4,1)],
                    [Sq(0,1,1) + Sq(1,3) + Sq(3,0,1) + Sq(4,2)]]
io values:  [[Sq(4)]]
io2 values:  [[Sq(1,1) + Sq(4)]]
Lift SqK exists:  True
SqK values:  [[1]]
```

Factoring the long exact cohomology sequences, we get short exact sequences defining H^*j and $H^*j/2$, with a map between them.

$$\begin{array}{ccccccccc} 0 & \leftarrow & K & \leftarrow & H^*j & \leftarrow & C & \leftarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \leftarrow & K_2 & \leftarrow & H^*j/2 & \leftarrow & C_2 & \leftarrow & 0 \end{array}$$

We next compute that

$$\mathrm{Ext}^1_{\mathcal{A}}(K, C) \cong \mathbf{F}_2 \cong \mathrm{Ext}^1_{\mathcal{A}}(K_2, C_2).$$

We then use the order of $\pi_7 j$ to show that j and $j/2$ determine the nontrivial extensions and compute them.

Proposition

$$\begin{aligned} H^*j/2 = \mathcal{A} \oplus \Sigma^7 \mathcal{A}/\mathcal{A}(&Sq^2\iota_0, Sq^{(0,1)}\iota_0, Sq^4\iota_0, \\ &Sq^{14}\iota_0 + Sq^{(4,1)}\iota_7, \\ &(Sq^{(0,1,1)} + Sq^{(3,0,1)} + Sq^{(1,3)} + Sq^{(4,2)})\iota_7). \end{aligned}$$

and

$$\begin{aligned} H^*j = \mathcal{A} \oplus \Sigma^7 \mathcal{A}/\mathcal{A}(&Sq^1\iota_0, Sq^2\iota_0, Sq^4\iota_0, \\ &Sq^8\iota_0 + Sq^1\iota_7, \\ &Sq^7\iota_7, \\ &(Sq^{(0,1,1)} + Sq^{(4,2)})\iota_7) \end{aligned}$$

The induced map is the evident quotient.

Note: these are defined over $\mathcal{A}(3)$ but not over $\mathcal{A}(2)$.

Resolving K

```
K = FP_Module([7],[[Sq(1)], [Sq(7)], [Sq(0,1,1) + Sq(4,2)]])

Kres0 = FP_Module(K.degs,[])
Kres1 = FP_Module(K.reldegs,[])

Keps = FP_Hom(Kres0,K,[[1]])
Kd0 = FP_Hom(Kres1,Kres0,K.rels)
Kres =[Keps,Kd0]

Kres = extend_resolution(Kres,2)
print "\n\nK resolution"
for j in range(len(Kres)):
    print j,": ",Kres[j].domain.degs
    print Kres[j].values
```

Computing Ext^1

```
print "\nCochains: "
for n in Kres[1].domain.degs:
    print "Degree ",n,": ",C[n]

print "\nCoboundaries: "
for n in Kres[0].domain.degs:
    print "Degree ",n,": ",C[n]

print "\nCocycles are in the kernel of d1 dual"
for ii in range(len(Kres[1].domain.degs)):
    dd = Kres[1].domain.degs[ii]
    print "Degree ",dd
    for x in C[dd]:
        print "Acting on ",x
        for c in [v.coeffs[ii] for v in Kres[2].values]:
            print (x*c).nf()
```

Results (cont.)

```
K resolution
0 :  [7]
[[1]]

1 :  [8, 14, 17]
[[Sq(1)],
 [Sq(7)],
 [Sq(0,1,1) + Sq(4,2)]]

2 :  [9, 15, 17, 18, 20, 21, 23]
[[Sq(1), 0, 0],
 [0, Sq(1), 0],
 [Sq(6,1), Sq(3), 0],
 [Sq(0,1,1), Sq(4), Sq(1)],
 [Sq(6,2), Sq(0,2), Sq(0,1)],
 [0, 0, Sq(4)],
 [Sq(2,2,1), 0, Sq(0,2)]]
```

Results (cont.)

```
Cochains:
Degree  8 :  [[Sq(8)]]
Degree  14 :  [[Sq(0,0,2)]]
Degree  17 :  []

Coboundaries:
Degree  7 :  []

Cocycles are in the kernel of d1 dual
Degree  8
Acting on  [Sq(8)]
[0]
[0]
[0]
[0]
[0]
[0]
[0]
```

Results (cont.)

```
Degree  14
Acting on  [Sq(0,0,2)]
[0]
[Sq(0,0,0,1)]
[0]
[0]
[0]
[0]
[0]

Degree  17
```

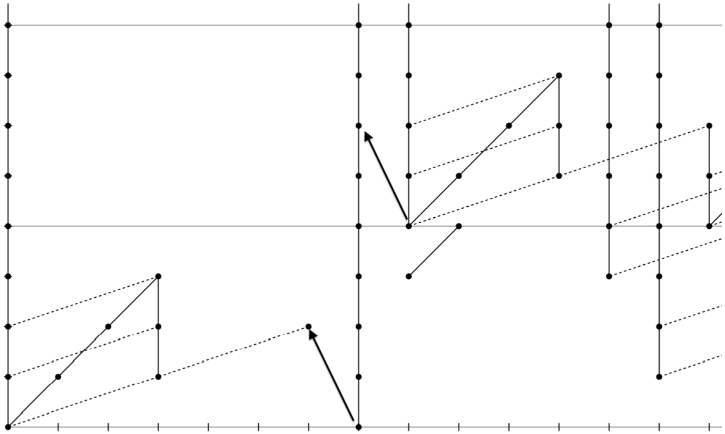
The nontrivial extension

- There is one nonzero cocycle, which sends ι_8 to Sq^8 and the other two generators to 0.
- If H^*j gave the split extension, then the E_2 term of the Adams spectral sequence converging to π_*j would be as shown on the next slide, where we have shown two d_2 differentials.
- The first d_2 must exist because $\nu^2 = 0$ in π_*j. The other is the first possible differential into the 7-stem. The result would still have $\pi_7(j) = \mathbf{Z}/(32)$, which is too large.
- Hence,

$$0 \longleftarrow K \longleftarrow H^*j \longleftarrow C \longleftarrow 0$$

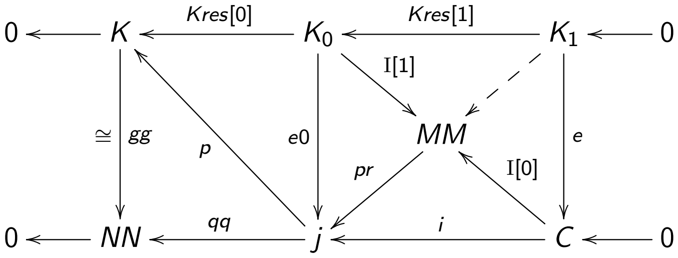
must be the nontrivial extension.

$\mathrm{Ext}_{\mathcal{A}}(K \oplus C, \mathbf{F}_2)$



The nontrivial extension

H^*j and the extension are computed as follows.



Results (cont.)

```
e = FP_Hom(Kres[1].domain,C,[[Sq(8)],0,0])

MM,I,P = DirectSum([C,Kres[0].domain])
j,pr = (I[1]*Kres[1] - I[0]*e).cokernel()
i = pr*I[0]
e0 = pr*I[1]
NN,qq = i.cokernel()
vv = [Kres[0].solve(g)[1] for g in Kres[0].codomain.gens()]
gg = FP_Hom(Kres[0].codomain,NN,[(qq(e0(x))).coeffs for x in vv])
cando,p = lift(qq,gg)

if not cando:
    print "Can't lift quotient map from j"

print "\n\nj.degs: ",j.degs
print "j.rels: ",j.rels
print "\ni: C --> j: ",i.values
print "\np: j --> K: ",p.values
```

Results (cont.)

```
sage: load j3.py
j.degs:  [0, 7]
j.rels:  [[Sq(1), 0],
          [Sq(2), 0],
          [Sq(4), 0],
          [Sq(8), Sq(1)],
          [0, Sq(7)],
          [0, Sq(0,1,1) + Sq(4,2)]]

i: C --> j:  [[1, 0]]

p: j --> K:  [[0], [1]]
```

$H^*j/2$ and the induced maps

It is helpful to break the transition from $H^*j/2$ to H^*j into two steps, as follows:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & K & \longleftarrow & H^*j & \longleftarrow & C & \longleftarrow & 0 \\
 & & \uparrow{\scriptstyle qK} & & \uparrow & & \| & & \\
0 & \longleftarrow & K_2 & \longleftarrow & J' & \longleftarrow & C & \longleftarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle qC} & & \\
0 & \longleftarrow & K_2 & \longleftarrow & H^*j/2 & \longleftarrow & C_2 & \longleftarrow & 0
\end{array}$$

The three extension cocycles map to one another,

$$e(H^*j) \mapsto e(J') \leftarrow\!\shortmid e(H^*j/2)$$

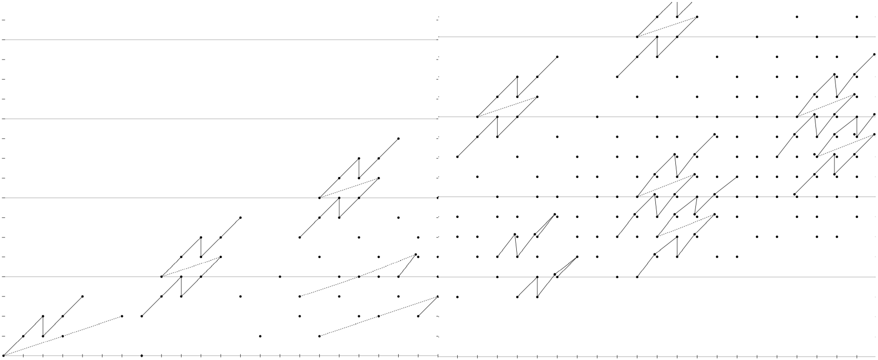
under the induced maps

$$\mathrm{Ext}^1_{\mathcal{A}}(K, C) \xrightarrow{qK^*} \mathrm{Ext}^1_{\mathcal{A}}(K_2, C) \xleftarrow{qC_*} \mathrm{Ext}^1_{\mathcal{A}}(K_2, C_2).$$

- Both qK^* qnd qC_* are isomorphisms, so that $H^*j/2$ also defines the nontrivial extension.
- The four modules K, K_2, C and C_2 are cyclic, and the maps qK and qC are the identity on the generating classes.
- It is then easily verified that $H^*j/2 \longrightarrow H^*j$ must send $\iota_0 \mapsto \iota_0$ and $\iota_7 \mapsto \iota_7$.

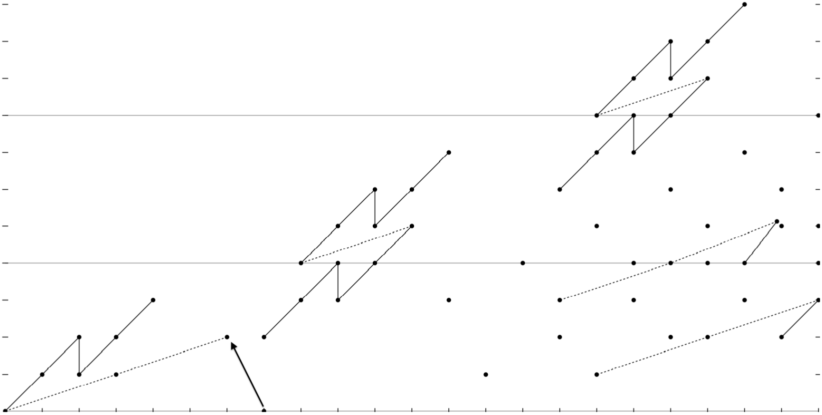
$\mathrm{Ext}_{\mathcal{A}}(H^* j/2, \mathbf{F}_2)$

We now run the ext code on the module $H^* j/2$. Through the 44 stem we get



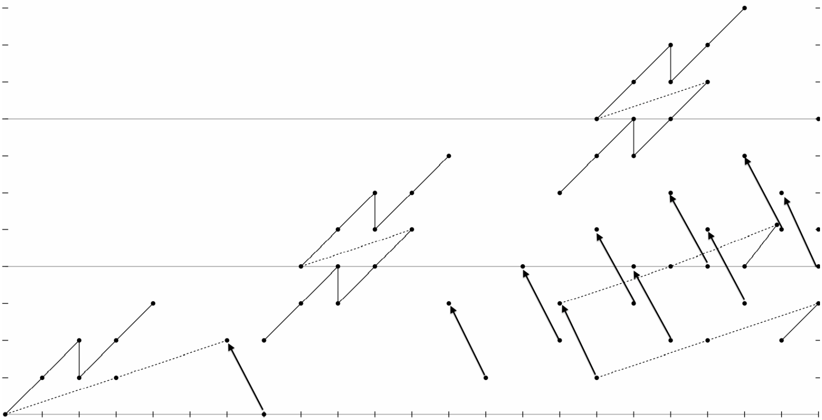
$d_2(\iota_7) = h_2^2$

There is an obvious d_2 which kills ν^2.



$E_3 = E_\infty$

The remarkable thing is that all the differentials are d_2s.



$E_3 = E_\infty$

Definition

Let ΩM denote the kernel of a minimal homomorphism from a free module onto M.

This is well defined for Frobenius algebras like $\mathcal{A}(n)$.

Remark

Since $\mathcal{A}$ is free as an $\mathcal{A}(n)$-module, tensoring up from $\mathcal{A}(n)$-Mod to $\mathcal{A}$-Mod is exact. Since $\mathcal{A}(n)$ is finite, calculations involving finitely presented $\mathcal{A}(n)$-modules are finite. These are the facts which allow the fpmods package to work.

Proposition

There is an epimorphism $\Omega^2 C_2 \longrightarrow K_2$. The kernel F exhibits 'Bott periodicity', $\Omega^4 F = \Sigma^{12} F$, and $\mathrm{Ext}_{\mathcal{A}}(F, \mathbf{F}_2)$ is the E_∞ term of the Adams spectral sequence for $\pi_* j/2$ in the range $s \geq 2$.

sage code to compute $\Omega^2 C_2$

```
C2 = FP_Module([0],  [[Sq(2)], [Sq(0,1)], [Sq(4)]],algebra=A3)

C2res0 = FP_Module(C2.degs,[])
C2res1 = FP_Module(C2.reldegs,[])
C2d0 = FP_Hom(C2res1,C2res0,C2.rels)
L2C2,i2C2 =C2d0.kernel()
print "\nLoops^2 C2:  degrees: ",L2C2.degs
print "rels: ",L2C2.rels

SK2 = FP_Module([8],[[Sq(4,1)], [Sq(0,1,1) + Sq(1,3) + Sq(3,0,1) + S
ff2 = FP_Hom(L2C2,SK2,[0,0,[1],0])

F2,inf2 = ff2.kernel()

print "\nKernel of Loops^2 C2 --> Susp K2, degrees: ",F2.degs
print "rels: ",F2.rels
```

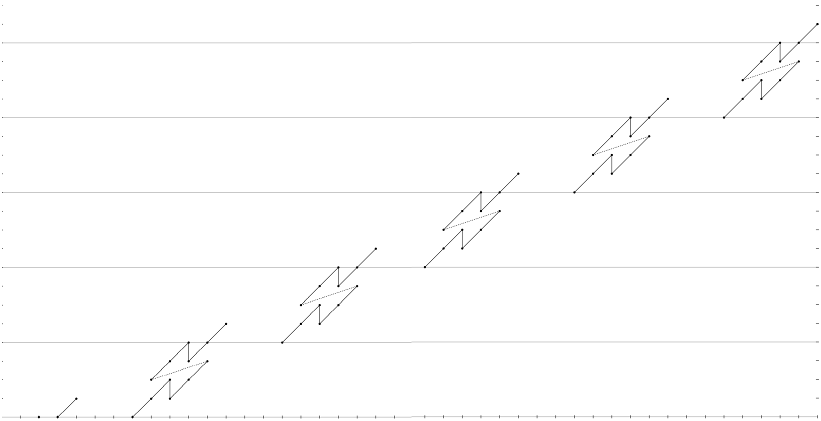
$\ker(\Omega^2 C_2 \longrightarrow \Sigma K_2$

```
sage: load j4.py

Loops^2 C2:  degrees:  [4, 5, 8, 9]
rels:  [[Sq(0,1), Sq(2), 0, 0],
 [Sq(0,0,1) + Sq(1,2) + Sq(7), 0, 0, Sq(2)],
 [Sq(1,0,1) + Sq(2,2), Sq(0,0,1) + Sq(1,2) + Sq(4,1), 0, Sq(0,1)],
 [0, Sq(0,1,1) + Sq(4,2), Sq(4,1), Sq(0,2)],
 [Sq(1,2,1), Sq(0,2,1) + Sq(4,3),
             Sq(0,1,1) + Sq(1,3) + Sq(3,0,1) + Sq(4,2), 0]]

Kernel of Loops^2 C2 --> Susp K2, degrees:  [4, 5, 9]
rels:  [[Sq(0,1), Sq(2), 0],
 [Sq(0,0,1) + Sq(1,2) + Sq(7), 0, Sq(2)],
 [Sq(1,0,1) + Sq(2,2), Sq(0,0,1) + Sq(1,2) + Sq(4,1), Sq(0,1)]]
```

Ext of the kernel



Periodic resolution of the kernel F

$$F_0 = \Sigma^4 \mathcal{A} \oplus \Sigma^5 \mathcal{A} \oplus \Sigma^9 \mathcal{A}$$

$$\uparrow \begin{bmatrix} Q_1 & Q_2 + Sq^{(1,2)} + Sq^7 & Sq^1 Q_2 + Sq^{(2,2)} \\ Sq^2 & 0 & Q_2 + Sq^{(1,2)} + Sq^{(4,1)} \\ 0 & Sq^2 & Q_1 \end{bmatrix}$$

$$F_1 = \Sigma^7 \mathcal{A} \oplus \Sigma^{11} \mathcal{A} \oplus \Sigma^{12} \mathcal{A}$$

$$\uparrow \begin{bmatrix} Sq^{(2,1)} & Sq^{(0,2)} + Sq^{(6)} & Q_1 + Sq^{(4,1)} + Sq^{(7)} \\ 0 & Sq^2 & Q_1 \\ 0 & Sq^1 & Sq^2 \end{bmatrix}$$

$$F_2 = \Sigma^{12} \mathcal{A} \oplus \Sigma^{13} \mathcal{A} \oplus \Sigma^{14} \mathcal{A}$$

Periodic resolution of the kernel F (cont.)

$$F_2 = \Sigma^{12}\mathcal{A} \oplus \Sigma^{13}\mathcal{A} \oplus \Sigma^{14}\mathcal{A}$$

$$\uparrow \begin{bmatrix} Sq^2 & Q_1 & Sq^4 \\ 0 & 0 & Q_1 \\ 0 & 0 & Sq^2 \end{bmatrix}$$

$$F_3 = \Sigma^{14}\mathcal{A} \oplus \Sigma^{15}\mathcal{A} \oplus \Sigma^{16}\mathcal{A}$$

$$\uparrow \begin{bmatrix} Sq^2 & Q_1 & Q_2 \\ Sq^1 & Sq^2 & Sq^6 \\ 0 & 0 & Sq^{(2,1)} \end{bmatrix}$$

$$F_4 \cong \Sigma^{12}F_0 = \Sigma^{16}\mathcal{A} \oplus \Sigma^{17}\mathcal{A} \oplus \Sigma^{21}\mathcal{A}$$

Proof that $E_3 = E_\infty$

If we let $K_2 \longleftarrow P_*$ and $F \longleftarrow F_*$ be free resolutions, then we have a free resolution

$$0 \longleftarrow \Omega^2 C_2 \longleftarrow F_0 \oplus P_0 \longleftarrow F_1 \oplus P_1 \longleftarrow \cdots$$

and therefore a free resolution of $H^* j/2$

$$0 \longleftarrow H^* j/2 \longleftarrow P_0 \oplus C_0 \longleftarrow P_1 \oplus C_1 \longleftarrow P_2 \oplus F_0 \oplus P_0 \longleftarrow P_3 \oplus F_1 \oplus P_1 \longleftarrow \cdots$$

The d_2 then cancels all but C_0, C_1, and the resolution of F. □

Remark

This can all be viewed as a consequence of the relation $\nu^2 = 0$.

The integral case is somewhat more complicated. We still have

Proposition

There is an epimorphism $\Omega^2 C \longrightarrow K$. The kernel F exhibits 'Bott periodicity' modulo h_0 towers.

Presentation of F:

$$F_0 = \Sigma^2\mathcal{A} \oplus \Sigma^4\mathcal{A} \oplus \Sigma^5\mathcal{A} \oplus \Sigma^9\mathcal{A}$$

$$\Big\uparrow \begin{bmatrix} Sq^1 & Sq^4 & 0 & Sq^{(2,0,1)} + Sq^{(6,1)} \\ 0 & Sq^2 & 0 & Q_2 + Sq^{(1,2)} + Sq^{(4,1)} \\ 0 & Sq^1 & 0 & Sq^6 \\ 0 & 0 & Sq^1 & Sq^2 \end{bmatrix}$$

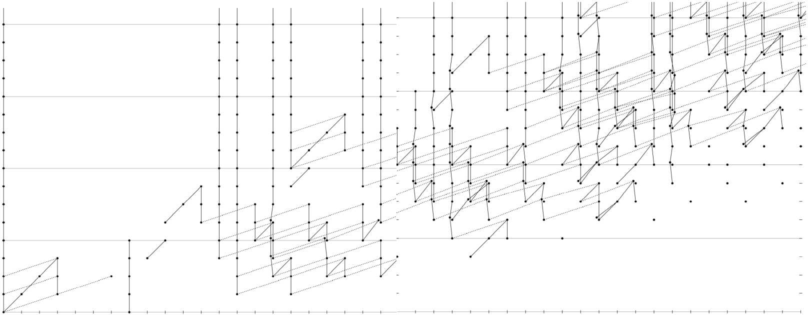
$$F_1 = \Sigma^3\mathcal{A} \oplus \Sigma^6\mathcal{A} \oplus \Sigma^{10}\mathcal{A} \oplus \Sigma^{11}\mathcal{A}$$

The d_2 coming from the epimorphism $\Omega^2 C \longrightarrow K$ cancels almost all of $\mathrm{Ext}_{\mathcal{A}}(H^* j, \mathbf{F}_2)$, leaving only the eta multiples and $\mathbf{Z}/8$s which persist to E_∞, together with adjacent towers every eight dimensions.
These towers already started to cancel with a $d_1(\beta)$ truncating the tower generated by σ. The general formula

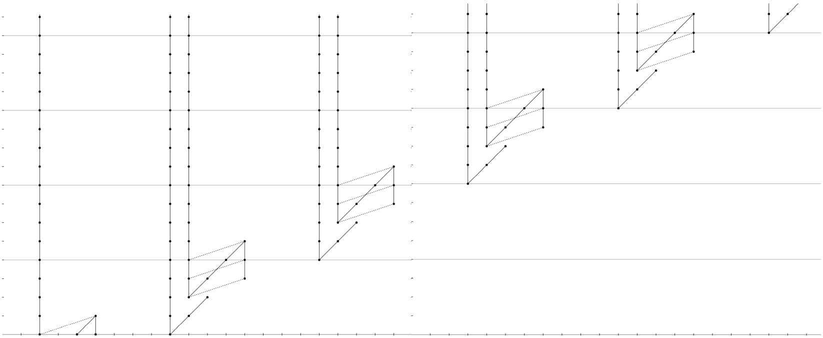
$$d_{r+1}(x^2) = h_0 x d_r(x)$$

now truncates the remaining towers in the well-known 2-adic pattern.

$\mathrm{Ext}_{\mathcal{A}}(H^* j, \mathbf{F}_2)$



$\mathrm{Ext}_{\mathcal{A}}(F, \mathbf{F}_2)$

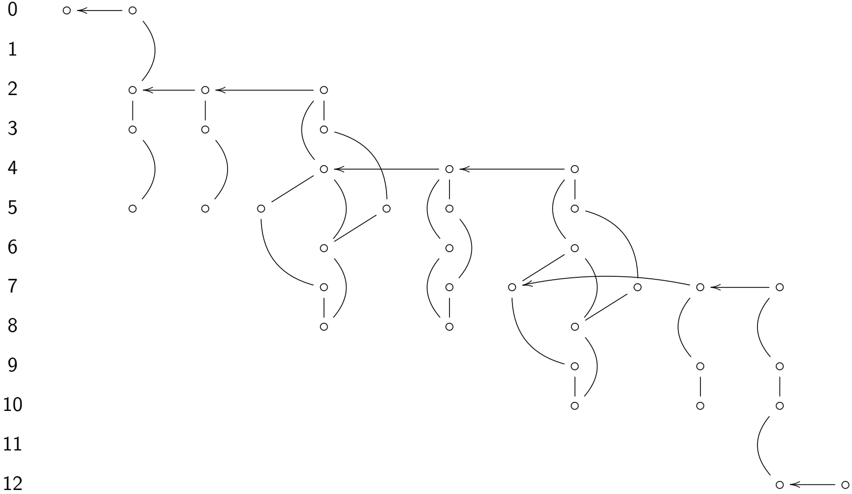


v_1^4

As a warm-up exercise, let us do such a calculation one chromatic level down.

We will compute a relative projective resolution of $H^*ko = \mathcal{A}//\mathcal{A}(1)$, where our relative projective class consists of modules extended up from $\mathcal{A}(0)$-Mod.

It is simplest to work in $\mathcal{A}(1)$-Mod. Here, we want a relative projective resolution of $\mathbf{F}_2$. This is easy and classical:


0
1
2
3
4
5
6
7
8
9
10
11
12

We get a couple of interesting consequences from this.

- Tensoring such relative projectives with Q_0-acyclic modules gives free modules. It follows that any Q_0-acyclic $\mathcal{A}(1)$-module M satisfies $\Omega^4 M \simeq \Sigma^{12} M$. E.G., $M = H^* RP^\infty$ or a mod 2 Moore space.
- It computes $\mathrm{Ext}_{A(1)}(\mathbf{F}_2, \mathbf{F}_2)$ for us:
  - The $\mathcal{A}(1)//\mathcal{A}(0)$ at the start gives an h_0-tower starting in $(0, 0)$.
  - The $\mathcal{A}(1)//\mathcal{A}(0)$ at the end gives an h_0-tower starting in $(s, t - s) = (3, 4)$.
  - The free modules give $\mathbf{F}_2$s in degrees $(1, 1)$ and $(2, 2)$.
  - Periodicity gives the rest.

It represents v_1^4 when restricted to $\mathrm{Ext}_{E(1)}(\mathbf{F}_2, \mathbf{F}_2)$.

v_2^8

Davis and Mahowald, in their 1982 paper "Ext over the subalgebra A_2 of the Steenrod algebra for stunted projective spaces" produce an analogous sequence representing v_2^8.

With the fpmods package, it is easy to experiment with alternatives. A somewhat simplified version of their sequence emerges from these calculations.

$\mathcal{A}(2)$ has many more subalgebras, with respect to which we can consider relative projectives.

An alternative perspective is that we can choose which Exts we consider "known".

Each such sequence results in a spectral sequence from 8 easier Ext modules to the cohomology of $\mathcal{A}(2)$ analogous to the Postnikov tower for *ko*.

Sequence representing v_2^8

For brevity, let us write $A = \mathcal{A}(2)$.

$$\mathbf{F}_2$$

$$\uparrow$$

$$A/(Sq^1, Sq^2)$$

$$\uparrow$$

$$\Sigma^4 A/(Sq^1, Sq^2 Sq^3)$$

$$\uparrow$$

$$\Sigma^8 A/(Sq^1)$$

$$\uparrow$$

$$\Sigma^{15} A \oplus \Sigma^{18} A/((Sq^1, 0), (Sq^3, 0), (Sq^4, Sq^1), (Sq^4 Sq^2, Sq^3))$$

(cont.)

The first half was the same as in Davis and Mahowald. This half is substantially smaller.

$$\Sigma^{22}A \oplus \Sigma^{24}A/((Sq^1, 0), (Q_1, Sq^1), (0, Q_1))$$

$$\uparrow$$

$$\Sigma^{26}A \oplus \Sigma^{30}A/((Sq^1), 0), (0, Sq^2))$$

$$\uparrow$$

$$\Sigma^{33}A \oplus \Sigma^{36}A/((Sq^1, 0), (Q_1, 0), (0, Sq^{(0,2)}))$$

$$\uparrow$$

$$\Sigma^{39}A \oplus \Sigma^{39}A/((Sq^1, Sq^1), (0, Sq^2), (Q_1, 0), (Sq^{(0,2)}, 0))$$

$$\uparrow$$

$$\Sigma^{56}\mathbf{F}_2$$

Let $T = T^n$ be the n-dimensional torus.
Known:

Theorem (RRB and JPCG)

$$ku^*_{T^n} = ku^*[y_1, \overline{y}_1 \ldots, y_n, \overline{y}_n]/(vy_i\overline{y}_i = y_i + \overline{y}_i)$$

with $y_i = c_1^{ku}(t_i)$ *and* $\overline{y}_i = c_1^{ku}(t_i^{-1})$.

New:

Theorem

$$ko^*_T = (ku^*_T)^{C_2}$$

where the C_2 *action,* $\tau(v) = -v$ *and* $\tau(y_i) = -\overline{y}_i$*, is given by complex conjugation.*

If we want explicit generators and relations, let $\mathbf{N} = \{1, 2, \ldots, n\}$ and recall that $ko^*_{Sp(1)^n} = ko^*[z_1, \ldots, z_n]$. We get a presentation of ko^*_T using its $ko^*_{Sp(1)^n}$-module structure.

Theorem

$$ko^*_T \cong ko^*\langle 1\rangle \oplus \bigoplus_{\emptyset \neq I \subset \mathbf{N}} ko^*[z_{\min I}, \ldots, z_n] \underset{\mathrm{ko}^*}{\otimes} ku^*\langle z_I\rangle/(z_i - z_{\{i\}}(0))$$

Here, $ku^*\langle z\rangle$ is additively ku^* suspended by the degree of z, with the ko^*-module structure that ku^* has, and with "$z(i) = v^i z$".

Thank you