The Finiteness Conjecture

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Outline

We will focus entirely on the prime 2 in this talk.

Conjecture (The Finiteness Conjecture)

A Sq^0 family

$$\{x, Sq^0(x), Sq^0Sq^0(x), \ldots, (Sq^0)^n(x), \ldots\}$$

in $\mathrm{Ext}_{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2) \Longrightarrow \pi_* S$ detects only a finite number of non-zero homotopy classes.

Examples

- $\{h_0, h_1, h_2, h_3\}$ detect the Hopf invariant one maps 2, η, ν, and σ. All higher members of this family die at E_2 by the differential $d_2(h_{n+1}) = h_0 h_n^2$.
- $\{h_0^2, h_1^2, h_2^2, h_3^2, h_4^2, h_5^2\}$ and perhaps h_6^2 detect Kervaire invariant one maps. Hill, Hopkins and Ravenel have shown that the remaining h_n^2 cannot be permanent cycles, but we do not yet know the differentials which might kill them.
- $\{c_0, c_1\}$ detect $\epsilon \in \pi_8$ and $\epsilon_1 \in \pi_{19}$ while $d_2 c_i = h_0 f_{i-1}$ for $i \geq 2$.
- $d_2 f_0 = h_0^2 e_0$, f_1 survives to at least E_5, and $d_3 f_i = h_1 y_{i-1}$ for $i \geq 2$.
- $d_2 e_0 = c_0^2 = h_1^2 d_0$, $d_3 e_1 = h_1 t_0$ and $d_2 e_i = h_0 x_{i-1}$ for $i \geq 2$.
- $d_2 y_i = h_0 h_{i+3} r_i$ for all $i \geq 0$ (note $y_1 = h_4 Q_3$)

Remark

Minami called this conjecture '*The new Doomsday Conjecture*'. His papers

The iterated transfer analogue of the new doomsday conjecture,
Trans. AMS **351** (1999) 2325–2351

and

The Adams spectral sequence and the triple transfer,
Am J. Math. **117** (1995) 965–985

provide some evidence, using the transfer, that it is true.

- All members of a Sq^0-family lie in the same Adams filtration $\mathrm{Ext}^{s,*}$.
- Sq^0 is a ring homomorphism by the Cartan formula.
- The original *Doomsday Conjecture* was that each filtration of the Adams spectral sequence detects only a finite number of homotopy classes.
- *That* was definitively refuted by Mahowald's η_j family, detected by the elements $h_1 h_j \in \mathrm{Ext}^{2,2^j+2}$.
- However, these do not form a Sq^0-family, since

$$Sq^0(h_1 h_j) = h_2 h_{j+1}, Sq^0(h_2 h_{j+1}) = h_3 h_{j+2}, \ldots$$

- With a finite number of low dimensional exceptions, the only elements in Adams filtration 2 which could be non-zero permanent cycles are the h_j^2 and the $h_1 h_j$. Thus, the Sq^0-families generated by the $h_1 h_j$ obey the finiteness conjecture.
- The *generic* Adams differential is $d_2 Q^i x = h_0 Q^{i-1} x$ if i is even.

Goal

To explain why I have always suspected that the Finiteness Conjecture is true.

Goal

To show how these methods give easy proofs of some of the results on the behavior of the Kervaire invariant one elements.

To gain some perspective on the Finiteness Conjecture, some Corollaries are:

- There are only a finite number of Hopf invariant one maps.
- There are only a finite number of Kervaire invariant one maps.
- Red shift (in K-theory and homotopy theory) is complicated.

So, it is unlikely that we will have a proof in the near future.

Squaring operations

The cohomology of a cocommutative Hopf algebra, such as the Steenrod algebra, has natural operations

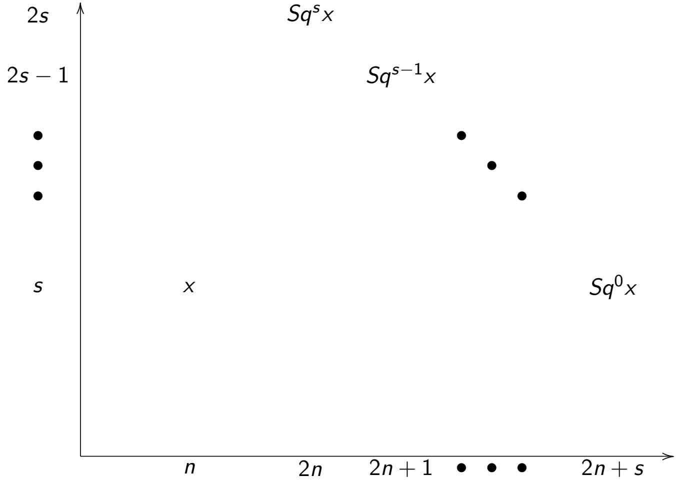
$$Sq^i : \mathrm{Ext}_A^{s,t}(\mathbf{F}_2, \mathbf{F}_2) \longrightarrow \mathrm{Ext}_A^{s+i,2t}(\mathbf{F}_2, \mathbf{F}_2)$$

for $0 \leq i \leq s$ in the cohomological indexing, or

$$Q^i : \mathrm{Ext}_A^{s,t}(\mathbf{F}_2, \mathbf{F}_2) \longrightarrow \mathrm{Ext}_A^{s+t-i,2t}(\mathbf{F}_2, \mathbf{F}_2)$$

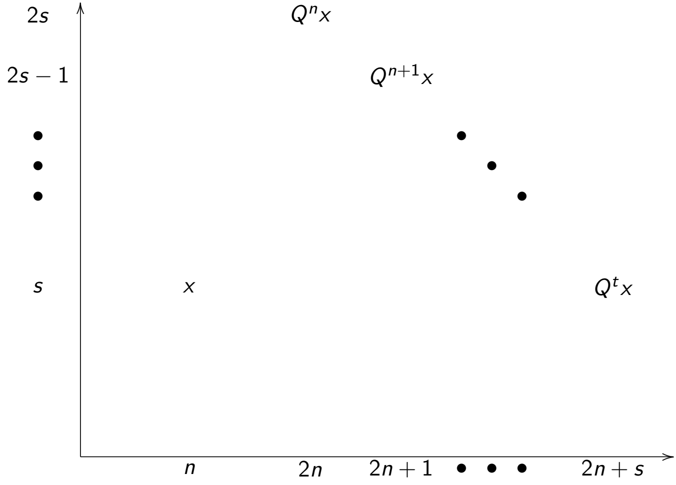
for $t - s \leq i \leq t$ in the homological indexing.

Cohomological indexing:



$$Sq^i : \mathrm{Ext}^{s,t} \longrightarrow \mathrm{Ext}^{s+i,2t} \quad (n = t - s)$$

Homological indexing:



$$Q^i : \mathrm{Ext}^{s,t} \longrightarrow \mathrm{Ext}^{s+t-i,2t} \quad (n = t - s)$$

S-algebra structure of the sphere

The product $\mu : S \wedge S \longrightarrow S$ factors through the homotopy orbits

$$\begin{tikzcd} S \wedge S \arrow[rr, "\mu"] \arrow[dr] & & S \\ & D_2 S := (S \wedge S)_{hC_2} \arrow[ur, "\xi"'] & \end{tikzcd}$$

Some notation:

- For $G \subset \Sigma_r$, $D_G X := (X^{\wedge r})_{hG}$
- Skeleta: $D_2^i X := S_+^i \times_{C_2} X \wedge X$ and $D_G^i X := EG_+^i \times_G X^{\wedge r}$
- Observe that $D_2^i S^n = \Sigma^n RP_n^{n+i}$, where $P_n^k = RP^k / RP^{n-1}$, the *stunted* projective space with cells in dimensions n through k.
- $P_n = P_n^\infty$.

Homotopy operations

$$S^n \xrightarrow{x} S$$

$$D_G S^n \xrightarrow{D_G x} D_G S \xrightarrow{\xi} S$$

Homotopy operations

$$S^n \xrightarrow{x} S$$

$$\begin{tikzcd} D_G S^n \arrow[r, "D_G x"] & D_G S \arrow[r, "\xi"] & S \\ S^k \arrow[u, "\alpha"] \arrow[urr, "\alpha^*(x)"'] & & \end{tikzcd}$$

Cup-i operations

We call the operation 'cup-i'

$$S^n \xrightarrow{x} S$$

$$\begin{tikzcd} D_2 S^n \arrow[r, "D_2 x"] & D_2 S \arrow[r, "\xi"] & S \\ S^{2n+i} \arrow[u, "\cup_i"] \arrow[urr, "\cup_i(x)"'] & & \end{tikzcd}$$

if

$$\begin{tikzcd} \cup_i \arrow[d, mapsto] & \in & \pi_{2n+i} D_2 S^n \arrow[d] \arrow[r, equal] & \pi_{2n+i} \Sigma^n P_n \\ \text{gen} & \in & H_{2n+i} D_2 S^n & \end{tikzcd}$$

Properties

- $\cup_0(x) = x^2$ and always exists.
- $\cup_i : \pi_n \longrightarrow \pi_{2n+i}$ is detected by Q^{n+i} in Ext
- Each cell of $D_2 S^n$ either defines a $\cup_i$ operation or a relation between lower operations.
- For example, $\cup_1 : \pi_n \longrightarrow \pi_{2n+1}$ exists iff n is even.
- If n is odd then the $2n+1$ cell of $D_2 S^n = \Sigma^n P_n$ instead gives a null-homotopy of $2x^2$.

Cup-1 of 2 is η

$$
\begin{tikzcd}
D_2 S \arrow[r, "D_2 2"] & D_2 S \arrow[r, "\xi"] & S \\
S^1 \arrow[u, "\cup_1"] \arrow[urr, "\eta"'] & &
\end{tikzcd}
$$

This is detected by $Sq^0(h_0) = h_1$ in Ext.

Cup-1 of η is not defined

$$\begin{tikzcd} D_2S^1 \arrow[r, "D_2\eta"] & D_2S \arrow[r, "\xi"] & S \\ S^3 \arrow[u, "\nexists \cup_1"] & & \end{tikzcd}$$

However, we do have $Sq^0(h_1) = h_2$ in Ext.
Restricting to the 3-skeleton,

$$\begin{tikzcd}
D_2^1 S^1 \arrow[r, "D_2\eta"] \arrow[d, "\mathrm{top}"'] & D_2^1 S \arrow[r, "\xi"] & S \\
S^3 \arrow[urr, "\nu"'] & &
\end{tikzcd}$$

The attaching map of the 3-cell of ΣP_1 has degree 2, and this gives an Adams spectral sequence differential $d_2(h_2) = h_0 h_1^2 = 0$, and there are no possible higher differentials, allowing ν to exist.

Similarly,

$$\begin{tikzcd} D_2^1 S^3 \arrow[r, "D_2\nu"] \arrow[d, "\mathrm{top}"'] & D_2^1 S \arrow[r, "\xi"] & S \\ S^7 \arrow[urr, "\sigma"'] & & \end{tikzcd}$$

Again, the attaching map has degree 2, and this gives $d_2(h_3) = h_0 h_2^2 = 0$, and there are no possible higher differentials, allowing σ to exist as well.

- After this, the differential $d_2(h_{n+1}) = h_0 h_n^2 \neq 0$, and no higher Hopf maps exist.
- In this sense, η must exist, while ν and σ are 'gifts', or low dimensional accidents.
- The 15 cell carrying h_4 is a null-homotopy of $2\sigma^2$, showing that $2\theta_3 = 0$.
- For higher n, we don't get the implication $2\theta_n = 0$ from the differential $d_2(h_{n+1}) = h_0 h_n^2$, though, because h_n was not a homotopy class to start with and the story is a bit more complicated.
- The boundary of the cell carrying h_n decomposes into a part carrying $h_0 h_n^2$ and a part carrying operations on $h_0 h_{n-1}^2$, effectively setting $2\theta_n$ equal to higher Adams filtration elements which we must analyze.

Operations on relations

If $2x = 0$ we can extend and operate on the extension as before

$$S^n \cup_2 e^{n+1} \xrightarrow{\quad \overline{x} \quad} S$$

$$D_G(S^n \cup_2 e^{n+1}) \xrightarrow{D_G\overline{x}} D_G S \xrightarrow{\quad \xi \quad} S$$

Operations on relations

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$$\begin{tikzcd}
D_G(S^n \cup_2 e^{n+1}) \arrow[r, "D_G\overline{x}"] & D_G S \arrow[r, "\xi"] & S \\
S^k \arrow[u, "\alpha"] \arrow[urr, "\alpha^*(\overline{x})"'] & &
\end{tikzcd}$$

To talk about $\pi_* D_2(X)$ and $H_* D_2(X)$:

- for a space X, $\Sigma^\infty QX \simeq \bigvee_r \Sigma^\infty D_{\Sigma_r} X$
- (suppress Σ^∞ henceforth)
- $H_* QX$ is the free Dyer-Lashof module on $H_* X$
- $H_* D_r X$ is the summand of weight r, where
    - $wt(H_* X) = 1$
    - $wt(ab) = wt(a) + wt(b)$
    - $wt(Q^i(a)) = 2wt(a)$
- the Nishida relations tell us the $\mathcal{A}$-module structure of $H_* QX$.

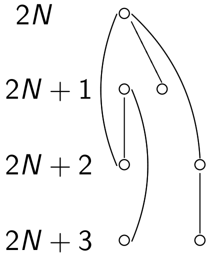
As an application,

Theorem

If θ_{n-1} exists, has order 2 and square 0, then θ_n exists and has order 2.

Proof: Let $N = 2^n - 2$ and let $X = S^N \cup_2 e^{N+1}$ and let $x \in H_N X$, $y \in H_{N+1} X$ be the generators. Let $\overline{\theta} : X \longrightarrow S$ be an extension of θ_{n-1} by a nullhomotopy of $2\theta_{n-1}$.

The bottom 4 dimensions of H_*D_2X and the attaching maps are shown at left, together with their images under $\xi D_2\theta_{n-1}$ and the detecting elements in the Adams spectral sequence on the right.

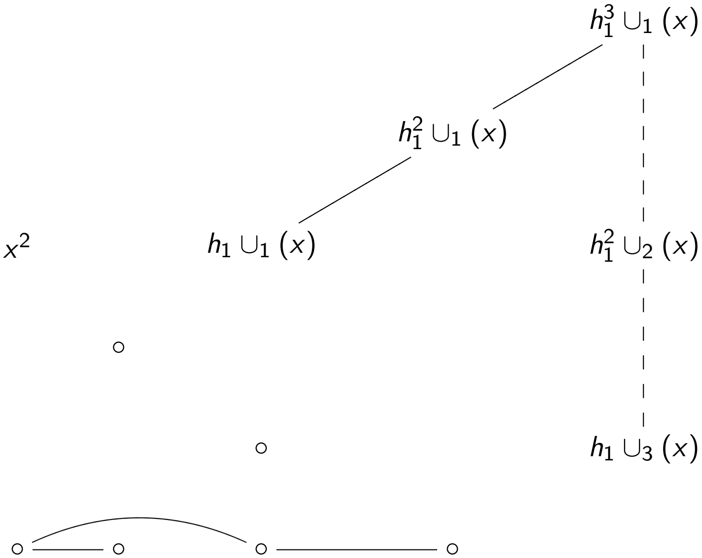


x^2	θ_{n-1}^2	$h_{n-1}^4 = 0$
$Q^{N+1}x$ xy	$\cup_1(\theta_{n-1})$ -	$Q^{N+1}(h_{n-1}^2) = 0$ $h_{n-1}^2 h_n = 0$
$Q^{N+2}x$ y^2	θ_n θ_n	h_n^2 h_n^2
$Q^{N+3}x$ $Q^{N+2}y$	- -	0 0

The assumption that $\theta_{n-1}^2 = 0$ means that $D_2X \xrightarrow{\bar{\theta}} S$ factors through the quotient by the bottom cell. The cell y^2 is then unattached and gives θ_n, while the cell $Q^{N+2}y$ gives a nullhomotopy of $2\theta_n$. $\square$

Other operations like $h_1 \cup_1$

There are many other operations than the $\cup_i$. For example, if $n = 3 \pmod 4$, there is an indecomposable homotopy operation $'h_1 \cup_1'$ detected by $h_1 Q^{n+1}$ in the Adams spectral sequence. This operation obeys the relation $2'h_1 \cup_1' (x) = 0$ and $\eta^2 `h_1 \cup_1' (x) = 4`h_1 \cup_3' (x)$.


$h_1^3 \cup_1 (x)$
$h_1^2 \cup_1 (x)$
x^2
$h_1 \cup_1 (x)$
$h_1^2 \cup_2 (x)$
$h_1 \cup_3 (x)$

Iteration

- Let $x_0 := x$ and $x_n = Sq^0(x_{n-1})$ be a Sq^0-family.
- Suppose $x : S^{t-s} \longrightarrow S$ has Adams filtration s.
- Then x_1 'lives' on the top cell of $\Sigma^{t-s} P_{t-s}^{t}$, so lies in the $2t - s$ stem.
- Similarly, x_2 'lives' on the top cell of $\Sigma^{2t-s} P_{2t-s}^{2t}$, so lies in the $4t - s$ stem.
- In general, x_{i+1} is carried by the top cell of $\Sigma^{2^i t-s} P_{2^i t-s}^{2^i t}$.
- Thus, the stems in which the x_i lie are converging to $-s$ 2-adically,
- while the cell of projective space on which x_i is carried is converging to 0 2-adically.
- The 0 cell of $P_{-\infty}^{\infty}$ is attached to every cell below it: $Sq^i(x^{-i}) = x^0$, so that
- for large i, x_i will only survive if all s obstructions vanish:
- $h_0 Sq^1(x_{i-1})$,
- $h_1 Sq^2(x_{i-1})$,
- $\langle h_1, h_0, Sq^3(x_{i-1}) \rangle$,
- $h_2 Sq^4(x_{i-1})$, …, down to the obstruction involving $Sq^s(x_{i-1}) = x_{i-1}^2$.

$$S^{t-s} \xrightarrow{x_0} S$$

$$\begin{tikzcd}
D_2^s S^{t-s} \arrow[r, "D_2 x_0"] \arrow[d] & S \\
S^{2t-s} \arrow[ur, "x_1"'] \arrow[u, dashed, bend left]
\end{tikzcd}$$

$$\begin{tikzcd}
D_2^s S^{2t-s} \arrow[r, "D_2 x_1"] \arrow[d] & S \\
S^{4t-s} \arrow[ur, "x_2"'] \arrow[u, dashed, bend left]
\end{tikzcd}$$

$$\begin{tikzcd} D_2^s S^{t-s} \arrow[r, "D_2 x_0"] \arrow[d] & S \\ S^{2t-s} \arrow[u, dashed, bend left] \arrow[ur, "x_1"'] & \end{tikzcd}$$

$$\begin{tikzcd} D_2^s S^{2t-s} \arrow[r, "D_2 x_1"] \arrow[d] & S \\ S^{4t-s} \arrow[u, dashed, bend left] \arrow[ur, "x_2"'] & \end{tikzcd}$$

$$\begin{tikzcd} D_2^s S^{4t-s} \arrow[r, "D_2 x_2"] \arrow[d] & S \\ S^{8t-s} \arrow[u, dashed, bend left] \arrow[ur, "x_3"'] & \end{tikzcd}$$

etc.

Iteration improved

Rather than apply D_2 only to x_i to get x_{i+1}, we could apply it to the whole triangle above, and use the natural map $D_2 D_2 \longrightarrow D_4$:

$$\begin{tikzcd}
D_2^s D_2^s S^{t-s} \arrow[rr] \arrow[d] & & D_4^{3s} S^{t-s} \arrow[r] \arrow[dd, "\mathrm{top}"] & S \\
D_2^s S^{2t-s} \arrow[u, dashed, bend left] \arrow[d] & & & \\
S^{4t-s} & & \bigvee S^{4t-s} \arrow[ll] \arrow[dl] & \\
& \bigvee_k S^{4t-s} \arrow[ul] \arrow[uuur, dashed, bend left] & &
\end{tikzcd}$$

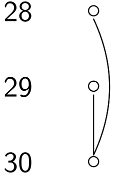
The classes we could operate upon to reach the Kervaire class in dimension 30 are

$$\begin{tikzcd}
h_0^2 \arrow[rr, "Q^2"] & & h_1^2 \arrow[rr, "Q^4"] & & h_2^2 \arrow[rr, "Q^8"] & & h_3^2 \arrow[rr, "Q^{16}"] & & h_4^2 \\
h_0 \arrow[u, "Q^0"] \arrow[r, "Q^1"'] & h_1 \arrow[ur, "Q^1"] \arrow[rr, "Q^2"'] & & h_2 \arrow[ur, "Q^3"] \arrow[rr, "Q^4"'] & & h_3 \arrow[ur, "Q^7"] \arrow[rr, dashed, "Q^8"'] & & h_4 \arrow[ur, dashed, "Q^{15}"] &
\end{tikzcd}$$

The differential $d_2(h_4) = h_0 h_3^2$ means that we cannot use h_4 in any simple way.

$$\begin{tikzcd}
h_0^2 \arrow[rr, "Q^2"] & & h_1^2 \arrow[rr, "Q^4"] & & h_2^2 \arrow[rr, "Q^8"] & & h_3^2 \arrow[rr, "Q^{16}"] & & h_4^2 \\
h_0 \arrow[u, "Q^0"] \arrow[r, "Q^1"'] & h_1 \arrow[ur, "Q^1"] \arrow[rr, "Q^2"'] & & h_2 \arrow[ur, "Q^3"] \arrow[rr, "Q^4"'] & & h_3 \arrow[ur, "Q^7"] \arrow[rr, dashed, "Q^8"'] & & h_4 \arrow[ur, dashed, "Q^{15}"] &
\end{tikzcd}$$

Next simplest is $Q^{16}(h_3^2)$ or $\cup_2(\theta_3)$. This lives on the top (i.e., 30) cell of $D_2^2 S^{14} = \Sigma^{14} P_{14}^{16}$:



This shows that θ_4 enforces the relation $2 \cup_1 (\theta_3) + \eta\theta_3^2 = 0$. For it to be a homotopy class rather than a null-homotopy, this relation would have to have already been true before θ_4 arrived to enforce it. This sounds like metaphysics, but is really an extension problem. Precisely, Consider an Adams resolution of S.

$$
\begin{tikzcd}
 & S^{30} \arrow[ddr, dashed] & e^{30} \arrow[l] \arrow[d] & S^{29} \arrow[l, "\supset"'] \arrow[d] \arrow[ddrrr, dashed, bend left] & & & \\
 & & \Sigma^{14} P_{14}^{16} \arrow[d] & \Sigma^{14} P_{14}^{15} \arrow[l, "\supset"'] \arrow[d] & S^{28} \arrow[l, "\supset"'] \arrow[d] & & \\
S & Y_1 \arrow[l] & Y_2 \arrow[l] & Y_3 \arrow[l] & Y_4 \arrow[l] & \cdots \arrow[l] & * \arrow[l]
\end{tikzcd}
$$

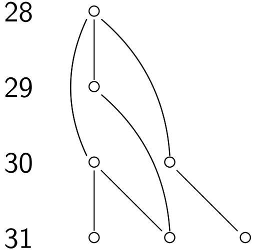
The 30-cell carrying θ_4 causes $2 \cup_1 (\theta_3) + \eta\theta_3^2$ to be 0 in $\pi_* Y_2$. It lives naturally in $\pi_* Y_3$, and if it were 0 there, it would factor through a point and θ_4 would exist.

$$
\begin{tikzcd}
h_0^2 \arrow[rr, "Q^2"] & & h_1^2 \arrow[rr, "Q^4"] & & h_2^2 \arrow[rr, "Q^8"] & & h_3^2 \arrow[rr, "Q^{16}"] & & h_4^2 \\
h_0 \arrow[u, "Q^0"] \arrow[r, "Q^1"'] & h_1 \arrow[ur, "Q^1"] \arrow[rr, "Q^2"'] & & h_2 \arrow[ur, "Q^3"] \arrow[rr, "Q^4"'] & & h_3 \arrow[ur, "Q^7"] \arrow[rr, dashed, "Q^8"'] & & h_4 \arrow[ur, dashed, "Q^{15}"] &
\end{tikzcd}
$$

Next consider $Q^{16}Q^7(h_3)$. Write classes in $H_*D_4S^7$ as follows:

- Q^i for the homology class $Q^i\iota_7$,
- Q^iQ^j for $Q^iQ^j\iota_7$, and
- $Q^i * Q^j$, etc., for their products.

Here are the bottom few cells of $D_4^2 S^7$.



ι_7^4	θ_3^2	$h_3^4 = 0$	5
$\iota_7^2 * Q^8$	-	$h_3^2 h_4 = 0$	4
$\iota_7^2 * Q^9$ $Q^8 * Q^8$	- θ_4	$h_3^2 Q^9 h_3 = 0$ h_4^2	4 2
$\iota_7^2 * Q^{10}$ $Q^8 * Q^9$ $Q^{16} Q^8$	- - -	$h_3^2 Q^{10} h_3 = 0$ $h_4 Q^9 h_3 = 0$ h_5	6 3 1

- $Q^8 * Q^8 + \iota_7^2 * Q^9$ is spherical,
- the operation it represents takes σ to θ_4,
- $Q^{16} Q^8 + \iota_7^2 * Q^{10}$ is attached by a map of degree 2 to $Q^8 * Q^8 + \iota_7^2 * Q^9$, so $2\theta_4 = 0$.

Manifold realization

- $D_4 S^7 = T(7\rho_4)$, the Thom complex of $7\rho_4$,
- Thom isomorphism $\Phi : H_* D_4 S^7 \longrightarrow H_* B\Sigma_4$, the weight 4 summand of $H_* QS^0$.

'*Extended powers of manifolds and the Adams spectral sequence*'
Contemp. Math., **271** (1999), 41–51

gives a dictionary

$$\{x \in H_* D_r S^n\} \xrightarrow{\Psi} \{M \xrightarrow{f} B\Sigma_r \mid f_*[M] = \Phi(x)\}$$

so that x is spherical iff $f^*(n\rho_r) = \nu_M$ and

$$(\alpha^* : \pi_n \longrightarrow \pi_k) \mapsto (N \mapsto \widetilde{\Psi(h(\alpha))} \times_{\Sigma_r} N^r).$$

$\Psi(Q^8 * Q^8 + \iota_7^2 Q^{10})$ is Jones'

$$S^1 \times S^1 \# RP^2 \longrightarrow B\Sigma_4$$

What have we done? We couldn't operate successfully on

$$\theta_3 = \sigma^2 : S^{14} \longrightarrow S$$

but by backing up a step, to $\sigma : S^7 \longrightarrow S$ we were successful:

$$\begin{tikzcd}
D_2S^{14} \arrow[rrr, bend left, "D_2\sigma^2"] \arrow[rr] & & D_4S^7 \arrow[r, "D_4\sigma"] & D_4S \arrow[r, two heads] & S \\
& S^{30} \arrow[ul, dashed, "\nexists"] \arrow[ur, "J"'] \arrow[urrr, "\theta_4"'] & & &
\end{tikzcd}$$

This suggests various strategies for higher Kervaire elements. Look for

$$\begin{tikzcd} D_8 S^7 \arrow[r, "D_8\sigma"] & D_8 S \arrow[r] & S \\ S^{62} \arrow[u, dashed, "\exists ?"] \arrow[urr, dashed, "\theta_5"'] & & \end{tikzcd}$$

$$\begin{tikzcd} D_{16} S^7 \arrow[r, "D_{16}\sigma"] & D_{16} S \arrow[r] & S \\ S^{126} \arrow[u, dashed, "\exists ?"] \arrow[urr, dashed, "\theta_6"'] & & \end{tikzcd}$$

but [Jones] there are no spherical classes which contain the classes needed to produce these elements. In fact, the attaching maps can be reduced to η, attaching to a cell carrying θ_4^2 or θ_5^2, resp., but no further.

There is another possibility. I thought for a bit that there might be a nice reverse symmetry, and we'd find

$$\begin{tikzcd} D_{16}S^3 \arrow[r, "D_{16}\nu"] & D_{16}S \arrow[r, two heads] & S \\ S^{62} \arrow[u, "\exists B"] \arrow[urr, "\theta_5"'] & & \end{tikzcd}$$

but

$$\begin{tikzcd} D_{32}S^3 \arrow[r, "D_{32}\nu"] & D_{32}S \arrow[r, two heads] & S \\ S^{126} \arrow[u, dashed, "\nexists"] \arrow[urr, dashed, "\theta_6"'] & & \end{tikzcd}$$

and finally

$$\begin{tikzcd} D_{64}S^1 \arrow[r, "D_{64}\eta"] & D_{64}S \arrow[r, two heads] & S \\ S^{126} \arrow[u, "\exists B'"] \arrow[urr, "\theta_6"'] & & \end{tikzcd}$$

But these meet exactly the same obstructions. (I am fairly certain.)

This strongly suggests

- $\eta\theta_n^2$ is the obstruction to θ_{n+1}
- $\eta\theta_n^2$ is 'accidentally' 0 in a couple more cases to allow the last few Kervaire classes, but
- $\eta\theta_6^2 \neq 0$ if θ_6 exists, or
- $\eta\theta_5^2 \neq 0$ if not.

- As we iterate Sq^0, we find that the number of obstructions which must cancel grows.
- Don Davis' results, saying that the h_i act monomorphically on initial segments of Ext in a range growing with i suggest that the 'stable obstruction'

$$h_0 Sq^1 + h_1 Sq^2 + < h_1, h_0, Sq^3 > + h_2 Sq^4 + \cdots$$

  will be nonzero 'generically'.
- Nishida's theorem tells us that the bottom cells of these large extended powers must map trivially, and it seems likely that this will extend some distance up from the bottom, setting up a race between nilpotence at the bottom and Sq^0 at the top of the large truncated extended powers.
- The root invariant is often detected by Sq^0 in the range we have seen, but if the Finiteness Conjecture holds, then this process must continually be interrupted as we iterate the root invariant.

The Bredon and Mahowald root invariants

Because of the connection with the root invariant, I want to show you the equivariant version of the root invariant.

Definition

Extend $x : S^n \longrightarrow S$ to $\overline{x} : S^{n+k\tau} \longrightarrow S$ with k maximal. The *Bredon root invariant*, $B(x)$ is then the underlying non-equivariant map $B(x) = U(\overline{x}) : S^{n+k} \longrightarrow S$.

- The cofiber sequence $C_{2+} \longrightarrow S \longrightarrow S^\tau$ smashed with $S^{n+k\tau}$ shows that the obstruction to extending one further is the composite

$$C_{2+} \wedge S^{n+k\tau} \longrightarrow S^{n+k\tau} \xrightarrow{\overline{x}} S$$

  which is the adjoint of $U(\overline{x})$, so $B(x)$ is always nonzero.
- As with Mahowald's root invariant, it is clearly a coset.
- Restricting $\overline{x}$ to fixed points gives x.

Theorem (Greenlees and B)

The Bredon root invariant equals the Mahowald root invariant.

See

'*The Bredon-Löffler conjecture*'
Experiment. Math. **4** (1995), no. 4, 289–297.

Theorem

The root invariant at least doubles the stem.

Proof:

$$\begin{tikzcd} S^n \arrow[r, "x"] \arrow[d, "\Delta"'] & S \\ S^n \wedge S^n \arrow[ur, "x \wedge x"'] & \end{tikzcd}$$

□

Theorem

$B(2) = \eta$, $B(\eta) = \nu$, $B(\nu) = \sigma$.

Proof: If D is one the the division algebras **R**, **C** or **H** then its double D' has an involution whose fixed point set is D. The Hopf construction on D' has the Hopf construction on D as its fixed points, and this extension is maximal. $\square$

Cartan formula

Theorem

Let $x_i \in \pi_{n_i} S^0$ and $B(x_i) \in \pi_{n_i+k_i} S^0$, for $i = 1, 2$. Let $k = k_1 + k_2$ and let $i : S^{-k-1} \longrightarrow P_{-k-1}$ be the inclusion of the bottom cell of the stunted projective space P_{-k-1}.

- *If $i_*(B(x_1)B(x_2)) \neq 0$ then $B(x_1)B(x_2) \subset B(x_1x_2)$.*
- *If $i_*(B(x_1)B(x_2)) = 0$ then $B(x_1x_2)$ lies in a higher stem than does $B(x_1)B(x_2)$.*

Proof: Certainly, the smash product of extensions of x_1 and x_2 is an extension of x_1x_2. The condition determines whether or not it is maximal. See

'*Some remarks on the root invariant*'
Stable and unstable homotopy (Toronto, ON, 1996),
Fields Inst. Commun. **19** 31–37

Thank you