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AN INFINITE FAMILY IN $\pi_* S^0$ DERIVED FROM MAHOWALD'S η_j FAMILY

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ABSTRACT. Combining the relationship due to D. S. Kahn between $\cup_i$ operations in homotopy and Steenrod operations in the E_2 term of the Adams spectral sequence with Mahowald's result that $h_1 h_j$ is a permanent cycle for $j \geqslant 4$, we show that $h_2 h_j^2$ is also a permanent cycle for $j \geqslant 5$. This gives another infinite family of nonzero elements in the stable homotopy of spheres. Properties of the $\cup_i$ homotopy operations further imply that these elements generate Z_2 direct summands.

Our objective is to prove the following theorem.

THEOREM. *For $j \geqslant 5$, $h_2 h_j^2$ is a permanent cycle in the* mod 2 *Adams spectral sequence of S^0. It detects a Z_2 direct summand of $\pi_n S^0$, $n = 1 + 2^{j+1}$.*

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Starting from Mahowald's result, that $h_1 h_j$ is a permanent cycle for all $j \geqslant 4$ [5], the proof is an easy application of the $\cup_i$ homotopy operations. We begin by defining them.

Let $D_2 X$ be the quadratic construction on X. That is, if X is a space and $(\ )^+$ denotes addition of a disjoint basepoint $+$, then $D_2 X = ((S^\infty)^+ \wedge X \wedge X)/Z_2$, where Z_2 acts by sending (r, x_1, x_2) to $(-r, x_2, x_1)$ and $(+, x_1, x_2)$ to $(+, x_2, x_1)$. If X is a spectrum then the construction of $D_2 X$ is more complicated. Details will be given in [2]. If $\Sigma^\infty X$ is the suspension spectrum of a space X then we have a natural isomorphism $D_2 \Sigma^\infty X \cong \Sigma^\infty D_2 X$ [2]. We will write as if we were using the spectrum construction for convenience (referring to $\pi_i S^0$ rather than $\pi_{i+n} S^n$, n large, for example), but the space level results of [3] suffice.

If $\alpha \in \pi_m D_2 S^n$ then α induces a homotopy operation $\alpha^*: \pi_n S^0 \to \pi_m S^0$ (where S^i is the i-sphere spectrum) as follows. For $x \in \pi_n S^0$, we let $\alpha^*(x)$ be the composite

$$S^m \xrightarrow{\alpha} D_2 S^n \xrightarrow{D_2 x} D_2 S^0 \xrightarrow{\xi} S^0$$

where $\xi = \Sigma^\infty \xi_1$: $D_2 S^0 \cong \Sigma^\infty (BZ_2)^+ \to \Sigma^\infty (S^0) = S^0$ is the map of spectra induced by the unique nontrivial map of based spaces ξ_1: $(BZ_2)^+ \to S^0$.

We point out in passing that α^* is not a homomorphism. In fact, $\alpha^*(x + y) = \alpha^*(x) + \tau(\alpha)xy + \alpha^*(y)$ where τ: $D_2 S^n \to S^{2n}$ is a spectrum level transfer map. The theory of these homotopy operations will be developed in [2].

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It is well known that $D_2S^n \cong \Sigma^n P_n$ where $P_n = RP^\infty/RP^{n-1}$ [3]. Let us also write $P_n^{n+i} = RP^{n+i}/RP^{n-1}$. Obstruction theory implies that if $S^0 = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$ is an Adams resolution for ordinary mod 2 homology, and if $x \in \pi_n S^0$ is represented by a map $S^n \to X_s$, then $\xi D_2 x$ induces a commutative diagram

$$\begin{array}{ccccccccccc} D_2S^n & = & \Sigma^n P_n & \supset & \Sigma^n P_n^{n+s} & \supset & \Sigma^n P_n^{n+s-1} & \cdots & \supset & \Sigma^n P_n^{n+1} & \supset & \Sigma P_n^n = S^{2n} \\ \downarrow & & & & \downarrow & & \downarrow & \cdots & & \downarrow & & \downarrow \\ S^0 & = & X_0 & \leftarrow & X_s & \leftarrow & X_{s+1} & \cdots & \leftarrow & X_{2s-1} & \leftarrow & X_{2s} \end{array}$$

[3, Proposition 4.2]. These maps send the characteristic maps of the top cells of each $\Sigma^n P_n^{n+i}$, $c_i \in \pi_{2n+i}(\Sigma^n P_n^{n+i}, \Sigma^n P_n^{n+i-1})$, to familiar algebraic constructions on the representative of x, $\bar{x} \in E_2^{s,n+s} = \mathrm{Ext}_A^{s,n+s}(Z_2, Z_2)$, where A is the mod 2 Steenrod algebra. Precisely, we have the following theorem.

THEOREM [3, THEOREM 4.4]. *The image of c_i in $E_2 = \mathrm{Ext}^{2s-i,2n+2s}$ is $\bar{x} \cup_i \bar{x}$.*

Several different notations have been used for $\bar{x} \cup_i \bar{x}$. We prefer to write $\mathrm{Sq}_i\bar{x}$ for $\bar{x} \cup_i \bar{x}$ and reserve $\cup_i$ for use in homotopy. The squaring operations here are those which apply to the cohomology $\mathrm{Ext}_A(M, N)$ of comodules M and N over a commutative Hopf algebra A (or, dually, modules M and N over a cocommutative Hopf algebra A) [4, §5], [6, §11]. In particular, if $\bar{x} \in \mathrm{Ext}^{s,n+s}$ then there are elements $\mathrm{Sq}_i\bar{x} \in \mathrm{Ext}^{2s-i,2n+2s}$ for $0 \leqslant i \leqslant s$, and $\mathrm{Sq}_0\bar{x} = \bar{x}^2$.

It is apparent then that the differentials on $\mathrm{Sq}_i\bar{x}$ are the successive lifts of the composite $S^{2n+i-1} \to \Sigma^n P_n^{n+i-1} \to X_{2s-i+1}$ of the n-fold suspension of the attaching map of the $n + i$ cell of P_n and the map induced by $\xi D_2 x$. In particular, when the attaching map is nullhomotopic, $\mathrm{Sq}_i\bar{x}$ is a permanent cycle. In addition, c_i can then be lifted to an element of $\pi_{2n+i}\Sigma^n P_n^{n+i}$ which defines a homotopy operation that we call $\cup_i\colon \pi_n \to \pi_{2n+i}$. Clearly $\cup_i(x)$ is detected by $\mathrm{Sq}_i\bar{x}$.

We are now ready to prove the theorem. Let x be Mahowald's η_j, detected by h_1h_j. Then $s = 2$ and $n = 2^j$. Computing Steenrod operations in H^*P_n shows that $\Sigma^n P_n^{n+2} = S^{2n} \vee (S^{2n+1} \cup_2 e^{2n+2})$. Thus $\cup_0(\eta_j) = \eta_j^2$ and $\cup_1(\eta_j)$ are defined but $\cup_2(\eta_j)$ is not. The attaching map of the $2n + 2$ cell shows that $2 \cup_1(\eta_j) = 0$. The corresponding elements in the mod 2 Adams spectral sequence are

$$\begin{aligned} \mathrm{Sq}_0(h_1h_j) &= h_1^2h_j^2, \\ \mathrm{Sq}_1(h_1h_j) &= h_1^2h_{j+1} + h_2h_j^2, \quad \text{and} \\ \mathrm{Sq}_2(h_1h_j) &= h_2h_{j+1}. \end{aligned}$$

This is immediate from the Cartan formula and the formulas $\mathrm{Sq}_0(h_j) = h_j^2$ and $\mathrm{Sq}_1(h_j) = h_{j+1}$ [1, p. 36 and Theorem 2.5.1]. Therefore $h_1^2h_j^2$ and $h_1^2h_{j+1} + h_2h_j^2$ are permanent cycles while $d_2(h_2h_{j+1}) = h_0h_2h_j^2$. (This differential is also immediate from the Hopf invariant one differential $d_2h_j = h_0h_j^2$. The Hopf invariant one differential is in turn an immediate consequence of the above formulas and the fact that if m is odd then $P_m^{m+1} = S^m \cup_2 e^{m+1}$.) Since $h_1^2h_{j+1}$ is a permanent cycle detecting $\eta\eta_{j+1}$, $h_2h_j^2$ is a permanent cycle detecting $\tau_j = \cup_1(\eta_j) - \eta\eta_{j+1}$. It is known that $h_2h_j^2 \neq 0$ if $j > 4$ [1, Theorem 2.5.1]. Also $h_2h_j^2$ is not a boundary since there are no elements which can hit it. Thus τ_j is nonzero. Since $2\cup_1(\eta_j) = 0$ and $2\eta = 0$, τ_j has order 2. Since there are no elements of lower filtration in the $2n + 1 = 1 + 2^{j+1}$ stem, τ_j is not divisible by 2. It follows that τ_j generates a Z_2 direct summand of the $1 + 2^{j+1}$ stem.

Note that the differential $d_2(h_2 h_{j+1}) = h_0 h_2 h_j^2$ is, as usual, not sufficient to imply that $2\tau_j = 0$. For this, the factorization of $\cup_1(\eta_j)$ through $\Sigma^n P_{n+1}^{n+2} = S^{2n+1} \cup_2 e^{2n+2}$ is needed.

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