On stable homotopy equivalences

by

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A fundamental construction in the study of stable homotopy is the free infinite loop space generated by a space X. This is the colimit $QX = \varinjlim \Omega^n \Sigma^n X$. The i^{th} homotopy group of QX is canonically isomorphic to the i^{th} stable homotopy group of X. Thus, one may obtain stable information about X by obtaining topological results about QX. One such result is the Kahn-Priddy theorem [7]. In another direction, Kuhn conjectured in [8] that the homotopy type of QX determines the stable homotopy type of X. In this note we prove his conjecture for a finite CW-complex X; that is, we prove the following.

Theorem If X and Y are finite CW-complexes, then QX and QY are homotopy equivalent if and only if $\Sigma^n X$ and $\Sigma^n Y$ are homotopy equivalent for some sufficiently large integer n. $\square$

Of course, one direction of our theorem is obvious. A homotopy equivalence $\Sigma^n X \to \Sigma^n Y$ clearly induces a homotopy equivalence $QX \to QY$. The proof in the other direction has three steps. Here is a brief outline of it. We begin with the case where X and Y are connected and we prove a p-local version of our result for each prime p. In doing so, we use results of Wilkerson, [11], to first express the stable p-local homotopy types of X and Y as bouquets of prime retracts. The same sort of analysis is then carried out, p-locally, on QX and QY, through an appropriate range of dimensions. The assumption that QX and QY are homotopy equivalent then forces the stable prime retracts of $X_{(p)}$ to match up with those of $Y_{(p)}$.

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In the second step the spaces are still assumed to be connected. Here we assemble the p-local results of step 1 using Mislin's notion of the stable genus of a space. The key ingredient in this step is Zabrodsky's presentation of the genus of certain spaces in terms of their self maps. The third step handles the case when the spaces are not connected. Here the Segal conjecture is used to show that X and Y must have the same number of path components.

In the final section we consider the related question of when $\{Q\Sigma^n X\}$ is the only infinite delooping of QX among connective spectra. Examples are provided showing that QX does not deloop uniquely in general. We also give a result which implies 2-local uniqueness of deloopings in certain cases.

We thank Clarence Wilkerson for helpful comments on this project. Results of his, from [11], play a crucial role in our proof. We also thank the referee for improvements in the exposition.

The connected p-local case

In this section we collect the results needed to prove the following.

Theorem 1 Let X and Y be connected finite CW-complexes. If QX and QY are homotopy equivalent at a prime p then X and Y are stably homotopy equivalent at p. $\square$

We start with a well known observation. Let ι and ϵ be used generically to denote the unit and counit of the adjunction between Σ^n and Ω^n, for $1 \leq n \leq \infty$.

Lemma 2 If a space B is a retract of an n-fold suspension, where $n \geq 1$, then there is a lift θ, of ι over ϵ.

$$\begin{array}{ccc} & & \Sigma^n\Omega^n QB \\ & \overset{\theta}{\nearrow} & \downarrow \epsilon \\ B & \xrightarrow{\iota} & QB \end{array} \qquad \square$$

When splitting the suspensions of certain p-local spaces, we will need the following definition from [11], wherein it is referred to as “H^*-prime”.

Definition Let X be a 1-connected p-local space of finite type. Then X is said to be *prime* if for every self-map $f : X \to X$, either
i) f induces an isomorphism in mod-p homology, or
ii) for every n, there exists an m such that the m-fold iterate of f induces the zero map on $H_i(X; \mathbf{Z}_p)$ for $0 < i \leq n$. $\square$

We will use the following facts from [11].

Theorem 3 (Wilkerson)
i) Any finite dimensional 1-connected p-local co-H-space is equivalent to a wedge of prime spaces.
ii) If a 1-connected p-local space of finite type is equivalent to a wedge of primes, then the prime wedge summands are unique up to order.
iii) A prime space which is a retract of a wedge of 1-connected p-local spaces of finite type is a retract of one of the wedge summands. $\square$

Proofs of the p-local results

Proof of Lemma 2 Let $B \xrightarrow{f} \Sigma^n W \xrightarrow{g} B$ be maps such that $gf \simeq 1$. The map $\theta = (\Sigma^n\Omega^n Qg)(\Sigma^n\iota)f$ in the following diagram provides the needed factorization $\iota = \epsilon\theta$.

$$\begin{array}{ccccc} \Sigma^n W & \xrightarrow{\Sigma^n \iota} & \Sigma^n\Omega^n Q\Sigma^n W & \xrightarrow{\epsilon} & Q\Sigma^n W \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle \Sigma^n\Omega^n Qg} & & \downarrow{\scriptstyle Qg} \\ B & \xrightarrow{\theta} & \Sigma^n\Omega^n QB & \xrightarrow{\epsilon} & QB \end{array} \qquad \square$$

Proof of Theorem 1 Let $\psi : QX \longrightarrow QY$ be a homotopy equivalence where X and Y are connected finite complexes. Fix a prime p and henceforth assume that all nilpotent spaces in this proof have been localized at p. (Of course, X and Y are not necessarily nilpotent, but they will become so after one suspension.) We will not burden the notation with this p-local assumption. Let n denote a fixed integer greater than both $\dim(X)$ and $\dim(Y)$. By Theorem 3, $\Sigma^n X$ and $\Sigma^n Y$ decompose uniquely into wedges of prime retracts. Let W denote the subbouquet of all prime retracts common to both spaces and let X' and Y' denote the rest. That is, write

$$\Sigma^n X \;\simeq\; W \vee X' \qquad \text{and} \qquad \Sigma^n Y \;\simeq\; W \vee Y'$$

where X' and Y' are assumed to have no nontrivial prime retracts in common. Now consider the composition

$$\Sigma^n X \xrightarrow{\Sigma^n(\iota)_n} \Sigma^n(QX)_n \xrightarrow{\Sigma^n(\psi)_n} \Sigma^n(QY)_n \xrightarrow{r} \Sigma^n Y$$

where $(\;)_n$ denotes the n-skeleton or restriction to it. The last map is the restriction of a retraction of $\Sigma^\infty QY$ onto $\Sigma^\infty Y$, which exists by Kahn's theorem [5]. In the stable range this map exists on the space level. Since X and Y are connected, each of the maps in this sequence induces a homology isomorphism in the lowest degree in which reduced homology for these spaces does not vanish. Applying the same argument to the inverse of ψ, it follows that the prime wedge summands of $\Sigma^n X$ and $\Sigma^n Y$ of lowest connectivity must have their homology mapped isomorphically by these composites, and hence be common to both $\Sigma^n X$ and $\Sigma^n Y$. Assuming that X and Y are not contractible, it follows then that the subbouquet W is not empty, and that the connectivity of X' and of Y' is strictly greater than that of W. We may assume $\mathrm{conn}(X') \leq \mathrm{conn}(Y')$. Note that

$$\begin{aligned}
\Sigma^n QX &\simeq \Sigma^n \Omega^n Q \Sigma^n X \\
&\simeq \Sigma^n \Omega^n Q(W \vee X') \\
&\simeq \Sigma^n \Omega^n (QW \times QX') \\
&\simeq \Sigma^n \Omega^n QW \ \vee \ \Sigma^n \Omega^n QX' \ \vee \ \Sigma^n(\Omega^n QW \wedge \Omega^n QX').
\end{aligned}$$

Similarly,

$$\Sigma^n QY \ \simeq \ \Sigma^n \Omega^n QW \ \vee \ \Sigma^n \Omega^n QY' \ \vee \ \Sigma^n(\Omega^n QW \wedge \Omega^n QY').$$

Now compare the $2n$-skeletons of $\Sigma^n QX$ and $\Sigma^n QY$. By Theorem 3, they each split uniquely as a wedge of primes. Among these prime retracts consider only those whose dimension is less than $2n$. Since the restriction of $\Sigma^n \psi$ and its inverse are at least $(2n-1)$-equivalences on the skeleta in question it follows that there is a one-to-one correspondence between the primes of dimension less than $2n$ in $\Sigma^n (QX)_n$ and those in $\Sigma^n (QY)_n$. From the decomposition of $\Sigma^n QX$ and $\Sigma^n QY$, obtained in terms of the summand W, it follows that the primes of dimension less than $2n$ in

$$\Sigma^n (\Omega^n QX')_n \ \vee \ \Sigma^n (\Omega^n QW \wedge \Omega^n QX')_n \tag{1}$$

must coincide, up to order, with those in

$$\Sigma^n (\Omega^n QY')_n \ \vee \ \Sigma^n (\Omega^n QW \wedge \Omega^n QY')_n \tag{2}$$

Observe that the $2n$-skeleton of QX' is just X' because X' is at least $n+1$-connected. Also, $\dim(X') < 2n$ since $\dim(X) < n$. Thus, the map θ of Lemma 2 factors through $\Sigma^n(\Omega^n QX')_n$ and provides a splitting of $\Sigma^n(\Omega^n QX')_n$ into X' and another factor whose connectivity is higher than that of X'. (The latter claim follows by examining homology.) Similarly for Y'.

Now let P be a prime in X' of minimal connectivity. It has dimension less than $2n$ because X' does. It occurs in (1), and so it must also occur in (2). It is not a retract of Y', by construction. It does not occur in the complement of Y' in $\Sigma^n(\Omega^n QY')_n$ because that complement has connectivity higher than that of Y' and hence higher than that of P. The connectivity of the second wedge summand in (2) is easily seen to be

$$1 + \mathrm{conn}(W) - n + \mathrm{conn}(Y') \;>\; \mathrm{conn}(Y') \;\geq\; \mathrm{conn}(X') \;=\; \mathrm{conn}(P)$$

so P cannot occur here either. Thus, no such prime P exists and we have shown that $X' \simeq *$. Since the connectivity of Y' is no lower than that of X', we must have $Y' \simeq *$ as well. □

The connected integral case

The goal in this section is to prove the following.

Theorem 4 Let X and Y be connected finite CW-complexes. If QX is homotopy equivalent to QY then X is stably homotopy equivalent to Y.
□

The proof of this theorem involves the genus $\mathcal{G}(X)$, of a space X, as defined by Mislin [10]. Recall that if X is a nilpotent space of finite type then $\mathcal{G}(X)$ is the set of all homotopy types $[Y]$ where Y is a nilpotent space with finite type and $Y_{(p)} \simeq X_{(p)}$ for each prime p. According to Wilkerson [13], $\mathcal{G}(X)$ is a finite set when X is a simply connected finite complex. Since localization commutes with suspension, there is an obvious map from $\mathcal{G}(X)$ to $\mathcal{G}(\Sigma X)$. Define the stable genus of a finite complex X to be

$$\mathcal{G}_s(X) = \varinjlim \mathcal{G}(\Sigma^n X).$$

It is not difficult to see that the stable genus of a finite complex X can be identified with $\mathcal{G}(\Sigma^n X)$ for n sufficiently large.

Suppose now that $QX \simeq QY$. Then, of course, $QX_{(p)} \simeq QY_{(p)}$ for every prime p. By Theorem 1, $X_{(p)}$ is stably homotopy equivalent to $Y_{(p)}$ for every prime p, and so $Y \in \mathcal{G}_s(X)$. Therefore to prove Theorem 4 it suffices to prove the following lemma.

Lemma 5 If X is a finite complex, then the function

$$\Phi : \mathcal{G}_s(X) \longrightarrow \mathcal{G}(QX)$$

that sends $[Z]$ in $\mathcal{G}(\Sigma^n X)$ to $[\Omega^n QZ]$ in $\mathcal{G}(QX)$ is one-to-one. $\square$

Proof: To show that Φ is one-to-one, it suffices to show that the composite

$$\mathcal{G}_s(X) \xrightarrow{\Phi} \mathcal{G}(QX) \longrightarrow \mathcal{G}(QX^{(n)})$$

is injective for some n. Here $QX^{(n)}$ denotes the Postnikov approximation of QX through dimension n. It may be obtained by attaching cells to kill off the homotopy groups of QX in dimensions greater than n. The second map here sends a homotopy type $[Z]$ in $\mathcal{G}(QX)$ to its Postnikov approximation $[Z^{(n)}]$ in $\mathcal{G}(QX^{(n)})$.

Let W denote a connected H-space which has finite type and only finitely many nonzero homotopy groups. The main result in Zabrodsky's paper [14], is an exact sequence

$$\mathcal{E}_t(W) \xrightarrow{d} (\mathbf{Z}_t^* / \pm 1)^\ell \longrightarrow \mathcal{G}(W) \longrightarrow *$$

which is defined as follows. In the middle term, $\mathbf{Z}_t^*$ denotes the group of units in the ring of integers modulo t. The exponent ℓ is the number of positive degrees k, in which the module of indecomposables in rational cohomology,

$$QH^k(W; \mathbf{Q}) \neq 0.$$

If there is more than one such degree k, order them $k_1 < \cdots < k_\ell$. The number t depends upon the space W. The prime divisors of t include those primes p, for which there is p-torsion in either the homotopy groups of W, or in the integral homology groups of W through degree k_ℓ, or in the cokernel of the Hurewicz homomorphism, through the same range. Zabrodsky gives a description of the smallest possible exponents $\nu_p(t)$, as well. However, for our purposes the exact value of t is unimportant; indeed given any one choice of t, any integer multiple of it works equally well in this sequence.

The first term in the sequence, $\mathcal{E}_t(W)$, denotes the monoid (under composition) of homotopy classes of those self-maps of W, which are local equivalences at each prime divisor of t. The function d then assigns to each such map a sequence of determinants - or rather the image of such a sequence in the middle group. The j^{th} determinant here is that of the linear transformation on $QH^{k_j}(W;\mathbf{Q})$ induced by the map f. Zabrodsky shows that this image is a subgroup and that the quotient is isomorphic as an abelian group to $\mathcal{G}(W)$.

If W is a finite complex in the stable range (and hence a co-H-space), there is a similar presentation of $\mathcal{G}(W)$. This was first proved by Davis in [4]; a much shorter proof of this was subsequently given in [9]. Take n to be larger than $\dim(X)$ so that $\Sigma^n X$ is in the stable range.

The map Φ fits into a commutative diagram

$$\begin{array}{ccccccc}
\mathcal{E}_t(\Sigma^n X) & \xrightarrow{d'} & (\mathbf{Z}_t^*/\pm 1)^\ell & \longrightarrow & \mathcal{G}_s(X) & \longrightarrow & * \\
\big\downarrow & & \big\downarrow = & & \big\downarrow \Phi & & \\
\mathcal{E}_t(QX^{(n)}) & \xrightarrow{d} & (\mathbf{Z}_t^*/\pm 1)^\ell & \longrightarrow & \mathcal{G}(QX^{(n)}) & \longrightarrow & *
\end{array}$$

whose left hand side sends a self-map f of $\Sigma^n X$ to the self-map $(\Omega^n Q f)^{(n)}$ of $QX^{(n)}$. Recall that $QX^{(n)}$ denotes the Postnikov approximation for QX through dimension n. To show Φ is one-to-one it suffices to show that the image of d is contained in the image of d'. Thus, given a self map f, of $QX^{(n)}$, it suffices to produce a self map of $\Sigma^n X$ with the same determinant sequence as f, up to sign. As mentioned earlier, we may view $QX^{(n)}$ as a cell complex obtained by attaching to QX cells of dimension $n+2$ and higher and so there is an equivalence of skeleta

$$(QX^{(n)})_n \simeq (QX)_n.$$

There is also a retraction

$$\Sigma^n(QX)_n \xrightarrow{r} \Sigma^n X.$$

Therefore, given a self-map

$$QX^{(n)} \xrightarrow{f} QX^{(n)}$$

we can take the composition

$$\Sigma^n \left[X \xrightarrow{\iota} QX \longrightarrow QX^{(n)} \xrightarrow{f} QX^{(n)}\right]_n \xrightarrow{r} \Sigma^n X$$

It is straightforward to check that this map has, up to signs, the same determinant sequence as f. This completes the proof of Lemma 5.

The non-connected case

Suppose now that X has path components $X_0, \ldots, X_n$, where $n \geq 1$. It is known, (e.g., see [1]), that in this case,

$$QX \quad \simeq \quad QX_0 \times \cdots \times QX_n \times (QS^0)^n.$$

Let B denote the path component of QS^0 that contains the constant loop. It is well known that $QS^0 \simeq \mathbf{Z} \times B$. Thus each path component of QX has the homotopy type of

$$QX_0 \times \cdots \times QX_n \times B^n.$$

Assume now that $QX \simeq QY$ where Y is another finite complex. Obviously then, QY, and hence Y, are not connected either. Each path component of QX must be homotopy equivalent to a path component of QY. Thus we may assume that

$$QX_0 \times \cdots \times QX_n \times B^n \quad \simeq \quad QY_0 \times \cdots \times QY_m \times B^m \tag{3}$$

where each X_i and Y_i is a connected finite complex. We first claim that $n = m$. To see this, assume $m < n$ and take the p-completion of both sides of (3). Completions commute with products. Moreover, after completing we can cancel the p-completion of B^m from both sides. This follows using the unique factorization results of [11] for p-local H-spaces with only finitely many nonzero homotopy groups and the results of [12] which imply that a p-complete homotopy type is determined by its sequence of n-types. Thus we have

$$(QX_0 \times \cdots \times QX_n \times B^{n-m})\widehat{\ }_p \quad \simeq \quad (QY_0 \times \cdots \times QY_m)\widehat{\ }_p \tag{4}$$

Since each Y_i is a finite connected complex, the Segal conjecture (or rather its affirmation, [3]) implies that there are no essential maps from $B\mathbf{Z}_p$ into QY_i and hence there are no essential maps of $B\mathbf{Z}_p$ into the right hand side. The Segal conjecture [2] also asserts that there are many essential maps from $B\mathbf{Z}_p$ into B, and hence into the left hand side of (4). From this contradiction we conclude that $m = n$.

Return now to equation (3), localize at p, and take the Postnikov approximation of both sides, through dimension d, where d exceeds the dimensions of X and Y. Results from [11], essentially the Eckmann-Hilton dual of Theorem 3, quoted earlier, allow us to cancel $(B^n)^{(d)}$ from both sides. We are left with

$$(QX_0 \times \cdots \times QX_n)^{(d)} \quad \simeq \quad (QY_0 \times \cdots \times QY_n)^{(d)}. \tag{5}$$

[2]Of course, the Kahn-Priddy theorem could have been used here just as well.

Notice that the left side has the form $(Q(X_0 \vee \cdots \vee X_n))^{(d)}$ and similarly for the right side. Thus both sides have the form $(QW)^{(d)}$ where W is a connected complex. If we restrict to the d-skeleton of both sides of (5) and suspend d times, it follows from Theorem 1 that $\vee_i X_i$ and $\vee_i Y_i$ are stably p-equivalent. Hence these two bouquets are in the same stable genus and, by Lemma 5, it follows that they are stably homotopy equivalent. Since

$$\Sigma^k X \quad \simeq \quad (\bigvee_{i=0}^{n} \Sigma^k X_i) \vee (\bigvee^{n} S^k),$$

when $k \geq 1$, and a similar expression holds for $\Sigma^k Y$, it follows that X and Y are stably homotopy equivalent. $\square$

When does QX deloop uniquely ?

The results obtained so far could be regarded as a first step toward answering the question just raised. This general question seems harder to answer.

Example 6 The space QS^1 has at least two distinct connective deloopings; namely $\{QS^n\}$ and $\{K(\mathbf{Z}, n)\times B^{n-1}F\}$, where F is the fiber of the retraction $QS^1 \to S^1$.

The key property here is that the circle S^1 is an infinite loop space. However, this method will not produce many more examples, since Hubbuck has shown that a finite complex which is a homotopy commutative H-space, for example, an infinite loop space, must be a torus [6].

If we consider QX for infinite complexes X, the Kahn-Priddy theorem, [7], provides some further examples of non-uniqueness of deloopings.

Example 7 Let $\mathcal{S}_p$ be the symmetric group on p letters. The space $Q\Sigma B\mathcal{S}_p$ has at least two distinct connective deloopings, because it is p-equivalent (as spaces but not as infinite loop spaces) to the product $\widetilde{QS^1} \times F'$, where $\widetilde{QS^1}$ denotes the universal cover of QS^1 and F' is the fiber of a p-local infinite loop map $Q\Sigma B\mathcal{S}_p \to \widetilde{QS^1}$.

The next theorem shows, on the other hand, that there are many examples where QX admits precisely one infinite loop structure, provided X is a double suspension. We say that a connected space is *atomic* if any self map of it which induces an isomorphism on its first nonzero reduced integral homology group is a homotopy equivalence.

Theorem 8 Let X be a 1-connected, 2-local, stably atomic finite CW-complex. Then $Q\Sigma^2 X$ has only one connective infinite delooping. $\square$

Proof: It was shown in [2] that $Q\Sigma^2 X$ is atomic when X satisfies the hypothesis of this theorem. Now assume that Y is a connected infinite loop-space and that

$$g : Q\Sigma^2 X \longrightarrow Y$$

is a homotopy equivalence. We need to produce an infinite loop map from $Q\Sigma^2 X$ to Y that is also a homotopy equivalence. Extend the composition

$$\Sigma^2 X \xrightarrow{\iota} Q\Sigma^2 X \xrightarrow{g} Y$$

to an infinite loop map $\theta : Q\Sigma^2 X \longrightarrow Y$. The composite

$$Q\Sigma^2 X \xrightarrow{\theta} Y \xrightarrow{g^{-1}} Q\Sigma^2 X$$

is an isomorphism on bottom dimensional homology, and hence an equivalence since $Q\Sigma^2 X$ is atomic. Since g and $g^{-1}\theta$ are homotopy equivalences, so is θ. $\square$

In fact, calculations suggest the following conjecture.

Conjecture 9 Let X be a finite CW-complex. Then $Q\Sigma^2 X$ has only one connective infinite delooping. $\square$

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