`ext` and its uses

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eCHT seminar on machine computation in homotopy theory
30 September 2021

Outline

Thanks, et cetera

- Mark Mahowald and Don Davis, for many interesting applications and novel uses,
- John Greenlees, for application to the root invariant,
- Tyler Lawson, for the dualizeDef code,
- John Rognes, for numerous coding improvements, suggestions and requests, and for collaboration in producing the database we just posted,
- Dan Isaksen, for asking me to talk, among other things,
- Thanks are due to the NSF, the Trond Mohn foundation (formerly Bergen Research foundation), the Simons Foundation, and the Knut and Alice Wallenberg Foundation for support for various parts of the work here.

What ext is: the code

C code and Unix shell scripts to produce

- minimal resolutions of modules M over a connected augmented $\mathbb{F}_2$-algebra A,
- chain maps lifting cocycles, giving products, induced maps and some Toda brackets,
- chain map computing the cocommutative Hopf algebra $Sq^0 : \mathrm{Ext}_{\mathcal{A}}^{s,t} \longrightarrow \mathrm{Ext}_{\mathcal{A}}^{s,2t}$,
- TeX code using TikZ for Adams style charts, designed to be incorporated into your TeX documents,
- TeX code for stem by stem summaries,
- new module definitions from old (tensor product, duals, skeleta and coskeleta) with some common modules built in.

The code is compartmentalized, so that it is easy to add new algebras A. Currently, only the Steenrod algebra and $\mathcal{A}(2) = \langle Sq^1, Sq^2, Sq^4 \rangle$ are included.

Resolutions

Given a module M over a connected augmented algebra A, ext produces a minimal resolution

$$0 \longleftarrow M \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \cdots \xleftarrow{d_S} C_S$$

through a specified internal degree. By minimality,

$$\mathrm{Hom}_A^t(C_s, \mathbb{F}_2) = \mathrm{Hom}_A(C_s, \Sigma^t \mathbb{F}_2) \cong \mathrm{Ext}_A^{s,t}(M, \mathbb{F}_2) .$$

The module M is specified either by

- a *module definition file*, which gives an $\mathbb{F}_2$ basis and the action of the Sq^i, or
- a finite presentation $0 \longleftarrow M \longleftarrow C_0 \longleftarrow C_1$. If the presentation isn't minimal, there will be a small d_1 to compute 'by hand' to finish $\mathrm{Ext}_A^s(M, \mathbb{F}_2)$ for $s = 0, 1$.

Chain maps

Given resolutions

$$0 \longleftarrow M \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \cdots \xleftarrow{d_S} C_S$$

and

$$0 \longleftarrow N \xleftarrow{d_0} D_0 \xleftarrow{d_1} D_1 \xleftarrow{d_2} \cdots \xleftarrow{d_S} D_S\,,$$

a class $x \in \mathrm{Ext}_A^{s_0,t_0}(M,N)$ can be represented as an A-homomorphism

$$x : C_{s_0} \longrightarrow \Sigma^{t_0} N\,.$$

From this data (specified in a *map definition file*), `ext` lifts x to a chain map $\{C_{s_0+s} \longrightarrow \Sigma^{t_0} D_s\}_s$.

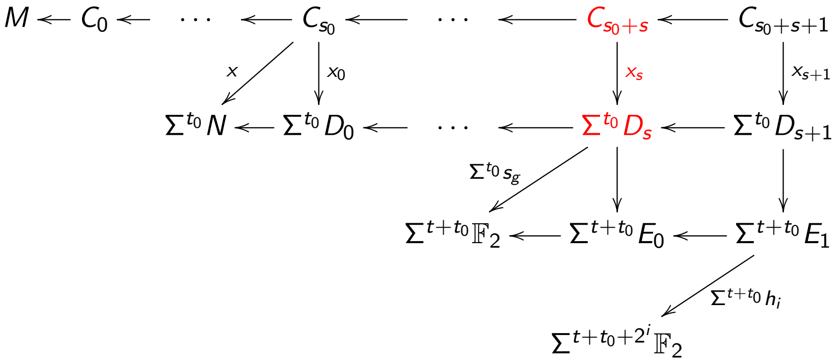
Chain maps

- The *map definition file* specifies s_0, t_0, M, N, x and the nonzero images in N of the A-generators of C_{s_0}.
- Running `newmap` on that map definition file will create the necessary files.
- Running `dolifts` in the domain directory will compute the lift.
- If $N = \mathbb{F}_2$ and $x = s_g$, dual to generator number g in C_s, the command `cocycle` will first create an appropriate map definition file and then run newmap.
- N.B.: the chain map is only computed in the range in which both domain and codomain have already been computed.
- `ext` contains a precomputed resolution of $\mathbb{F}_2$ over $\mathcal{A}$ through $t = 120$, and a resolution of $\mathbb{F}_2$ over $\mathcal{A}(2)$ through $t = 300$; this is usually all you need.

Chain maps, cont.

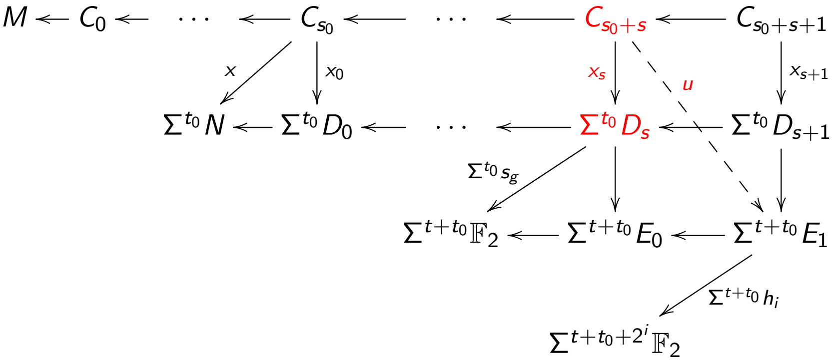
The chain map lifting $x \in \mathrm{Ext}^{s_0,t_0}(M,N)$ gives

- the product map $Ext^{s,t}(N,\mathbb{F}_2) \xrightarrow{\cdot x} Ext^{s+s_0,t+t_0}(M,\mathbb{F}_2)$ and
- Toda brackets $\langle h_i, s_g, x\rangle$.



Chain maps, cont.

Toda brackets $\langle h_i, s_g, x\rangle = h_i u + vx$ where $u : s_g x \simeq 0$ and $v : h_i s_g \simeq 0$.



A canonical basis for $\text{Ext}_{\mathcal{A}}$

- The $\mathcal{A}$-generators of C_s are indexed on the non-negative integers in non-decreasing order of internal degree.
- Their duals $s_0, s_1, \ldots$ form a well-defined canonical basis for Ext, which we now describe.
- In general, this depends on the linearly ordered basis of M, but for $M = \mathbb{F}_2$, is unique.
- This gives a canonical basis for the cohomology of the Steenrod algebra.
- We'll describe the case $M = \mathbb{F}_2$ here.

The order

Totally order the terms $Sq^R s_g^*$ of $C_{s,t}$ by

$$Sq^R s_g^* < Sq^{R'} s_{g'}^*$$

iff

1. $g < g'$, or
2. $g = g'$ and $Sq^R < Sq^{R'}$, where the Milnor basis elements Sq^R are given reverse lexicographic order: $(r_1, r_2, \ldots) < (r_1', r_2', \ldots)$ iff for some k, $r_k < r_k'$ and $r_i = r_i'$ for all $i > k$. (grevlex)
3. Thus,

$$\begin{aligned} &(n) < (n-3,1) < (n-6,2) < \cdots \\ &< (n{-}7,0,1) < (n{-}10,1,1) < \cdots < (n{-}14,0,2) < (n{-}17,1,2) < \cdots \\ &< (n-15,0,0,1) < (n-18,1,0,1) < \cdots < (n-22,0,1,1) < \cdots \end{aligned}$$

The order, cont.

- Each nonzero element $x \in C_{s,t}$ then has a *leading term* $\mathrm{LT}\, x$, which is the lowest term in x.
- In the totally ordered basis $\{Sq^R s_g^*\}$ of a given bidegree (s,t), the decomposable elements, those with $\deg(Sq^R) > 0$, form an initial segment which is followed by the $\mathcal{A}$-generators s_g^* of bidegree (s,t).

We can now inductively define our canonical basis as follows.

1. We start with the bases $\{0_0^*\}$ for C_0 and $\{\}$ for C_s with $s > 0$.
2. We may inductively assume given the basis for C_s in degrees less than t, and for C_{s-1} in degrees less than or equal to t.

genimker

Step 1: Generating the image and kernel.

- $\mathrm{Im}_{s,t}$ will be a totally ordered list of pairs (x, dx) with the leading terms of the dx in strictly increasing order. $\mathrm{Ker}_{s,t}$ will be a list of terms x. Both are initially empty.
- Consider the terms $Sq^R s_g^*$ in order. Let $x = Sq^R s_g^*$ and compute $dx = Sq^R d(s_g^*)$. Then, while $dx \neq 0$, if $LT(dx) = LT(dy)$ for a pair $(y, dy) \in \mathrm{Im}_{s,t}$, replace x by $x - y$ and dx by $dx - dy$. If not, add (x, dx) to $\mathrm{Im}_{s,t}$ and proceed to the next term. If, instead $dx = 0$, add x to the end of the list $\mathrm{Ker}_{s,t}$.
- Note that the leading term of dx will be increased each time we replace dx by $dx - dy$ until it either becomes 0 or has a leading term not already found among the dy in $\mathrm{Im}_{s,t}$.

addgen

Step 2: Adding new generators.

- We may inductively assume given $\text{Ker}_{s-1,t}$.
- For each $x \in \text{Ker}_{s-1,t}$, in order, let $c = x$.
- While $LT(x) = LT(dy)$ for some pair $(y, dy) \in \mathrm{Im}_{s,t}$, replace x by $x - dy$.
- If this process terminates with $LT(x) \neq 0$, add a new generator s_g^* with $d(s_g^*) = c$, then add a new pair (z, x) to $\mathrm{Im}_{s,t}$, where z is the difference of s_g^* and those y whose images dy were subtracted from c to get the final x with a new leading term.
- If the process terminates with $x = 0$, do nothing.

Remark

- We could choose, at this second step, to let $d(s_g^*) = x$ and add the pair (s_g^*, x) to $\mathrm{Im}_{s,t}$. The ext code prior to the year 2000 used that algorithm.
- Experience shows that the bases s_g obtained from the algorithm described here have products, especially the $h_i \cdot s_g$, monomial far more frequently than those produced by the old algorithm.
- The Wayne State Research Report #37 (1997) used the older algorithm. The first difference visible in Ext charts lies in bidegree $(9, 9+23)$. In the new algorithm,

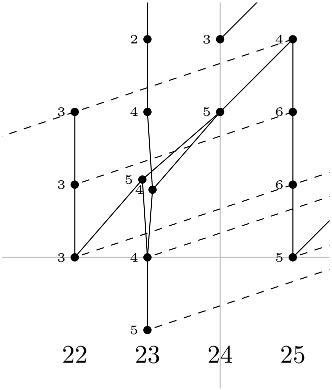
$$h_1 8_3 = 9_4 \qquad \text{and} \qquad h_0 8_4 = 9_5 \,.$$

  In the older algorithm,

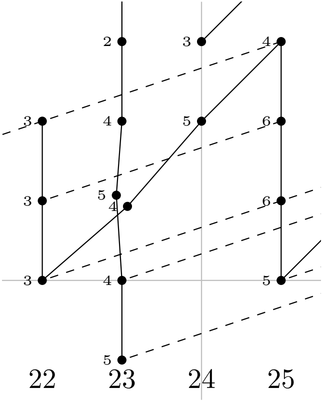
$$h_1 8_3 = 9_4 + 9_5 \qquad \text{and} \qquad h_0 8_4 = h_0 8_5 = 9_5 \,.$$

Old and new

Pre-2000



Current



Old and new

- The change alters the resolution much earlier, and can be seen by doing hand calculations in low degrees.
- In the old algorithm, $d(2_1^*) = Sq^{(0,1)} \cdot 1_0^* + Sq^2 \cdot 1_1^*$.
- In the new algorithm, $d(2_1^*) = Sq^3 \cdot 1_0^* + Sq^2 \cdot 1_1^*$.

The dataset

John Rognes and I have produced a dataset of our calculations of the cohomology of the Steenrod algebra which we have placed in a public repository for public use. It is available for download at the NIRD Research Data Archive

`https://archive.sigma2.no`.

The goal is to have a stable reference which does not depend on personal web pages or other changeable sources and to give it a DOI for citations or references.

The dataset, cont

Hence, please cite/refer to the dataset by its digital object identifier DOI:10.11582/2021.00077, viz.

Robert R. Bruner and John Rognes, *The cohomology of the mod 2 Steenrod algebra* (2021), https://doi.org/10.11582/2021.00077. [Dataset]. Norstore.

The DOI can also be used to access the dataset directly as

HTTPS://DOI.ORG/10.11582/2021.00077.

The dataset contents

Let $\mathcal{A}$ denote the classical mod 2 Steenrod algebra over $\mathbb{F}_2$.
The archive contains

1. a minimal resolution of $\mathbb{F}_2$ over $\mathcal{A}$ in internal degrees $t \leq 184$ and cohomological degrees $s \leq 128$,
2. chain maps lifting each member in the resulting basis for $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ in this range, and
3. a chain map which gives the Hopf algebra squaring operation

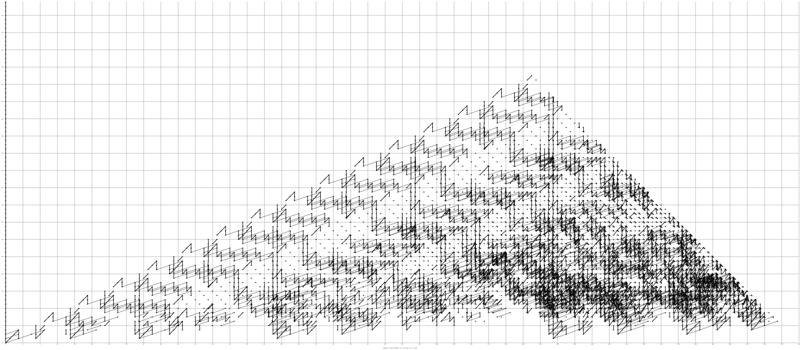
$$Sq^0 : \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathcal{A}}^{s,2t}(\mathbb{F}_2, \mathbb{F}_2).$$

Contents, continued

In addition to this data, the archive contains

1. A document *The cohomology of the mod 2 Steenrod algebra* (CohomA2.pdf),
2. version 1.9.3 of the `ext` package, used to produce this data,
3. a text file `all.products` containing all nonzero products,
4. a text file `all.sq0` containing all nonzero $Sq^0(x)$,
5. text files `P.txt`, `P2.txt` and `P4.txt` giving the Adams periodicity operators P, P^2, and P^4,
6. a text file `MM.txt` giving a variant of Isaksen's 'Mahowald operator' $M(x) = \langle g_2, h_0^3, x\rangle$,
7. an Adams chart `Ext-A-F2-F2-0-184.pdf`, and
8. a stem by stem summary `S-184.pdf`.

The chart



The document

The included document *'The cohomology of the mod 2 Steenrod algebra'* may be of independent interest and is available on arXiv as `arXiv:2109.13117`.

It contains

1. a description of the data files defining the resolution and the chain maps,
2. details on the operators P, P^2, P^4, M and M',
3. a description of the algorithm, as above, defining our canonical basis, and
4. a concordance relating our canonical basis to previous notations.

Operators P^i and M

These are defined by

1. $Px = \langle h_3, h_0^4, x\rangle$,
2. $P^2x = \langle h_4, h_0^8, x\rangle$,
3. $P^4x = \langle h_5, h_0^{16}, x\rangle$ and
4. $Mx = \langle g_2, h_0^3, x\rangle$.

Note that

5. Isaksen's Mahowald operator $Mx = \langle g_2, h_0^3, x\rangle$ is not of the form that it is immediately evident in our dataset,
6. its variant
$$M'(x) = \langle h_0, h_0^2 g_2, x\rangle$$
is.
7. Both contain $\langle h_0 g_2, h_0^2, x\rangle$, when defined.

Operators P^i and M', cont.

Chain maps allow us to calculate $\langle h_i, s_g, x \rangle$: recall

$$
\begin{array}{ccccccccccccc}
M & \leftarrow & C_0 & \leftarrow & \cdots & \leftarrow & C_{s_0} & \leftarrow & \cdots & \leftarrow & C_{s_0+s} & \leftarrow & C_{s_0+s+1} \\
 & & & & & \swarrow{\scriptstyle x} & \downarrow{\scriptstyle x_0} & & & & \downarrow{\scriptstyle x_s} & & \downarrow{\scriptstyle x_{s+1}} \\
 & & & & \Sigma^{t_0} N & \leftarrow & \Sigma^{t_0} D_0 & \leftarrow & \cdots & \leftarrow & \Sigma^{t_0} D_s & \leftarrow & \Sigma^{t_0} D_{s+1} \\
 & & & & & & & & & \swarrow{\scriptstyle \Sigma^{t_0} s_g} & \downarrow & & \downarrow \\
 & & & & & & & & \Sigma^{t+t_0}\mathbb{F}_2 & \leftarrow & \Sigma^{t+t_0} E_0 & \leftarrow & \Sigma^{t+t_0} E_1 \\
 & & & & & & & & & & & \swarrow{\scriptstyle \Sigma^{t+t_0} h_i} & \\
 & & & & & & & & & & \Sigma^{t+t_0+2^i}\mathbb{F}_2 & &
\end{array}
$$

Operators P^i and $M^\prime$, cont.

Proposition

If $h_i \cdot (s_0)_{g_0} = 0$ and $(s_0)_{g_0} \cdot x = 0$ then the Toda bracket $\langle h_i, (s_0)_{g_0}, x\rangle$ contains the sum of all those s_g such that the chain map lifting x, applied to s_g^*, contains a term $Sq^{2^i} \cdot (s_0)_{g_0}^*$.

In particular, it is entirely possible that

1. $x_s(s_g^*)$ contains $Sq^{2^i} \cdot (s_0)_{g_0}^*$, though
2. $h_i \cdot (s_0)_{g_0} \neq 0$ or $(s_0)_{g_0} \cdot x \neq 0$.

Operators P^i and M', cont.

The files `P.txt`, `P2.txt`, `P4.txt` and `MM.txt` contain the data needed to make these calculations. Each file starts with a short header describing the operator and the file's organization, then has three sections:

(a) values of the brackets,

(b) nonzero products which obstruct existence of the bracket, and

(c) nonzero products which give the indeterminacy.

In general, the indeterminacy in a bracket $\langle a, b, c\rangle$ is $a(\mathrm{Ext}) + (\mathrm{Ext})c$. However, the brackets P, P^2, P^4 and M' have indeterminacy $a(\mathrm{Ext})$ because $(\mathrm{Ext})c$ is contained in $a(\mathrm{Ext})$ in these cases.

Brackets in general

1. Tangora (1970) writes Px for a cycle represented by $(b_{02})^2 x$ in the May spectral sequence. This is justified by the differential $d_4(b_{02}^2) = h_0^4 h_3$.
2. However, this accounts for only part of the definition of a Massey product or Toda bracket.
3. If a, b and x satisfy $ab = bx = xa = 0$, the Jacobi identity says

$$0 \in \langle a, b, x \rangle + \langle b, x, a \rangle + \langle x, a, b \rangle \,.$$

   If A, U and V satisfy $d(A) = ab$, $d(U) = bx$ and $d(V) = xa = ax$, then
   - $Ax + aU \in \langle a, b, x \rangle$,
   - $Ua + bV \in \langle b, x, a \rangle$ and
   - $Ax + bV \in \langle b, a, x \rangle = \langle x, a, b \rangle$.
4. Approximating the bracket $\langle a, b, x \rangle$ by Ax, in those cases where Ax is a cycle, fails to distinguish between $\langle a, b, x \rangle$ and $\langle b, a, x \rangle = \langle x, a, b \rangle$. These differ by $\langle b, x, a \rangle = \langle a, x, b \rangle$.

Brackets in general, cont.

4. This can lead to greater indeterminacy and to anomalies like Tangora's observation (Note 3, p. 48) that
   - $h_5 i$ is annihilated by h_0^3, but
   - $P(h_5 i)$, if defined to be $(b_{02})^2 h_5 i$, has $h_0^9 \cdot (b_{02})^2 h_5 i \neq 0$.
   - In fact, consulting `P.txt` we see that $P(h_5 i) = P(8_{26}) = 0$ with zero indeterminacy.
5. By using only the precisely defined brackets we
   - get the advantages of their good formal behavior, and
   - limit the indeterminacy.

Other brackets

Two other operators may be of use. They are

1. the complex Bott periodicity operator

$$v_1(x) = \langle h_0, h_1, x \rangle ,$$

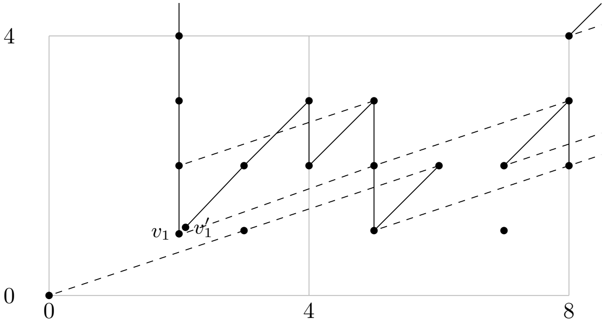
   and

2. the mod 2 'Bott periodicity operator'

$$v_1'(x) = \langle h_1, h_0, x \rangle .$$

3. In the universal example, the Adams spectral sequence for $\pi_*(S \cup_2 e^1 \cup_\eta e^2)$,
   - $v_1(0_0)$ is an h_1-annihilated class which supports an infinite h_0-tower, while
   - $v_1'(0_0)$ is an h_0-annihilated class which supports h_1^2-multiplication.

v_1 and v_1'



Concordance

We present the relation between

1. our s_g basis,
2. names based on Tangora's calculation of the E_∞ term of the May spectral sequence in 1970, and
3. Chen's Lambda algebra computation of Ext^s for $s \leq 5$ in 2011.

In the process,

4. we make a natural extension to traditional notation using Sq^0, and
5. adopt Isaksen's use of the Mahowald operator.

Indeterminacy

1. May spectral sequence names are inherently indeterminate; we are explicit about this.
2. We relate Chen's Lambda algebra names to our s_g by use of relations known in each description. Some of this requires Chen's unpublished description of the decomposable classes in Ext^6 from 2012.
3. A natural homomorphism $\Lambda \longrightarrow \mathrm{Hom}(C_*, \mathbb{F}_2)$, or equivalently, an action $\Lambda \otimes C_* \longrightarrow \mathbb{F}_2$ would be useful.
4. We define $f_0 = Sq^1(c_0)$ and $y = Sq^2(f_0)$. This eliminates the indeterminacy in May spectral sequence or Toda bracket definitions. Nassau, Tilson and I have calculated these using Nassau's method (arXiv:1909.03117v3).

$\mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2)$

- Shimada and Iwai showed that, in Henriques' notation,

$$\mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, h_1, h_2, c_0, d_0, e_0, \alpha, \beta, \gamma, \delta, g, w_1, w_2]/I$$

  where $I = (h_0 h_1, h_0^2 h_2 - h_1^3, h_1 h_2, \ldots)$ is an ideal generated by 54 relations.
- Rognes and RRB found that analyzing the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}(2)}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_* tmf$$

  was facilitated by working over the subalgebra $R_0 = \mathbb{F}_2[g, w_1, w_2]$.

Adams spectral sequence

1. The E_2-term is a sum of cyclic R_0-modules with annihilator ideals (0), (g) or (g^2). The sum is finite except for four h_0-towers. The differential d_2 is R_1-linear, where $R_1 = \mathbb{F}_2[g, w_1, w_2^2]$.
2. The E_3-term is a sum of cyclic R_1-modules and three non-cyclic summands, each with two generators. The differential d_3 is R_2-linear, where $R_2 = \mathbb{F}_2[g, w_1, w_2^4]$.
3. The E_4-term is a sum of cyclic R_2-modules and four non-cyclic summands, each with two generators. The differential d_4 is again R_2-linear.
4. The $E_5 = E_\infty$-term is a sum of cyclic R_2-modules and three non-cyclic summands, each with two generators.

Adams spectral sequence for $C2$, $C\eta$ and $C\nu$

1. We apply a similar strategy to the Adams spectral sequence for $tmf_*(C2)$, $tmf_*(C\eta)$ and $tmf_*(C\nu)$.
2. Write M_{2^k} for the two cell $\mathcal{A}(2)$-module with Sq^{2^k} nontrivial.
3. We have a long exact sequence

$$\cdots \xrightarrow{h_k} \mathrm{Ext}^{s,t}_{\mathcal{A}(2)} \xrightarrow{i} \mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(M_{2^k}, \mathbb{F}_2) \xrightarrow{j} \mathrm{Ext}^{s,t-2^k}_{\mathcal{A}(2)} \xrightarrow{h_k} \cdots,$$

   and hence,
4. a short exact sequence

$$0 \longrightarrow \mathrm{Cok}\, h_k \longrightarrow \mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(M_{2^k}, \mathbb{F}_2) \longrightarrow \mathrm{Ker}\, h_k \longrightarrow 0$$

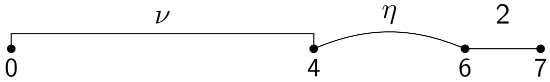
Adams spectral sequence for $C2$, $C\eta$ and $C\nu$, cont.

5. From the R_0-module description of $\mathrm{Ext}_{\mathcal{A}(2)}$, we write $\mathrm{Cok}\, h_k$ and $\mathrm{Ker}\, h_k$ as R_0-modules.
6. Using `ext` to compute chain maps inducing i and j, we can choose (good) lifts $\overline{x}$ for the R_0-module generators x of $\mathrm{Ker}\, h_k$, solving $j(\overline{x}) = x$.
7. Using `ext` to compute chain maps, we can compute the R_0-action on our lifts $\overline{x}$, and thereby resolve the extension questions in the short exact sequence above.
8. This gives $\mathrm{Ext}^{s,t}_{\mathcal{A}(2)}(M_{2^k}, \mathbb{F}_2)$ as an R_0-module.

Adams spectral sequence for $C2$, $C\eta$ and $C\nu$, cont.

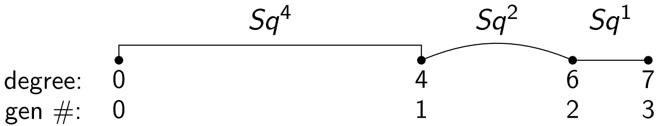
10. From this we can compute E_3 as an R_1-module, and E_4 and $E_5 = E_\infty$ as R_2-modules. (In fact $E_4 = E_\infty$ for $C\eta$.)
11. Although a Gröbner basis for $E_2 = \mathrm{Ext}_{\mathcal{A}(2)}$ was quite useful, writing the E_3, E_4 and E_5 terms as P/I, so that we could use Gröbner basis methods, was not. Using `ext` to compute in our canonical s_g basis turns out to be easier and more effective.
12. The same strategy works to compute tmf_* of the four cell complex

X =



A cofiber sequence

Let M be the cohomology of X:



This sits in a short exact sequence

$$0 \longrightarrow \Sigma^6 M_1 \xrightarrow{i} M \xrightarrow{j} M_4 \longrightarrow 0\,.$$

We will show how to create

1. the module definition file for M,
2. the map definition files for i and j, and
3. the 1-cocycle e defining the extension.

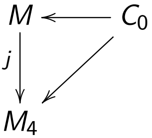
The obvious entries in the module definition file for M are

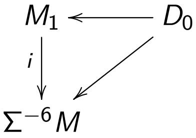
```
4
0 4 6 7

0 4 1 1
1 2 1 2
2 1 1 3
```

We will use the `ext` code to see what else is required.

The cocycles defining i and j are
(with minimal resolutions $M \longleftarrow C_*$ and $M_1 \longleftarrow D_*$)





with module definition files

```
0 0 M M4 j 1

0
1
0 0 1 x80
```

```
0 -6 M1 M i 1

0
1
2 0 1 x80
```

The extension cocyle $e \in \mathrm{Ext}_{\mathcal{A}}^{1,6}(M_4, M_1)$ is the E_2 representative of the connecting map $\Sigma^6 C2 \longrightarrow \Sigma C\nu$ in

$$C\nu \longrightarrow X \longrightarrow \Sigma^6 C2 \longrightarrow \Sigma C\nu$$

and sits in a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^6 M_1 & \longrightarrow & M & \longrightarrow & M_4 & \longrightarrow & 0 \\
\uparrow & & \uparrow e & & \uparrow & & \| & & \\
E_2 & \longrightarrow & E_1 & \xrightarrow{d} & E_0 & \longrightarrow & M_4 & \longrightarrow & 0
\end{array}$$

with $d(1_0^*) = Sq^1 0_0^*$, $d(1_1^*) = Sq^2 0_0^*$, $d(1_2^*) = Sq^6 0_0^*$, $\ldots$.

Then $e(1_2^*)$ is the degree 6 class in $\Sigma^6 M_1$, and
e sends all other generators of E_1 to 0.

The resulting map file for e is then

```
1 6 M4 M1 e 1

2
1
0 0 1 x80
```

These three chain maps then compute for us the long exact sequence

$$\mathrm{Ext}^{s,t}(M_4) \xrightarrow{j} \mathrm{Ext}^{s,t}(M) \xrightarrow{i} \mathrm{Ext}^{s,t-6}(M_1) \xrightarrow{e} \mathrm{Ext}^{s+1,t}(M_4)$$

(Retire to the terminal window.)

Summary

- ext is fast.
- ext gives access to products, brackets and induced maps.
- ext has created an extensive database of calculations relevant to the homotopy groups of spheres.
- ext is useful even in situations, like $\mathrm{Ext}_{\mathcal{A}(2)}(M, \mathbb{F}_2)$, where machine calculation isn't strictly needed.
- ext can quickly give useful low dimensional calculations.
- ext isn't all that user friendly.

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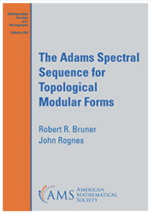

Mathematical Surveys and Monographs
Volume: 253; 2021; 690 pp; Hardcover
MSC: Primary 18; 55;

Print ISBN: 978-1-4704-5674-0
Product Code: SURV/253
List Price: $125.00
AMS Member Price: $100.00
MAA Member Price: $112.50

Not yet published

Expected publication date
October 14, 2021

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Electronic ISBN: 978-1-4704-6563-6
Product Code: SURV/253.E

The Adams Spectral Sequence for Topological Modular Forms

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The connective topological modular forms spectrum, tmf, is in a sense initial among elliptic spectra, and as such is an important link between the homotopy groups of spheres and modular forms. A primary goal of this volume is to give a complete account, with full proofs, of the homotopy of tmf and several tmf-module spectra by means of the classical Adams spectral sequence, thus verifying, correcting, and extending existing approaches. In the process, folklore results are made precise and generalized. Anderson and Brown-Comenetz duality, and the corresponding dualities in homotopy groups, are carefully proved. The volume also includes an account of the homotopy groups of spheres through degree 44, with complete proofs, except that the Adams conjecture is used without proof. Also presented are modern stable proofs of classical results which are hard to extract from the literature.

Tools used in this book include a multiplicative spectral sequence generalizing a construction of Davis and Mahowald, and computer software which computes the cohomology of modules over the Steenrod algebra and products therein. Techniques from commutative algebra are used to make the calculation precise and finite. The H_∞ ring structure of the sphere and of tmf are used to determine many differentials and relations.

Readership

Graduate students and researchers interested in algebraic topology, specifically in stable homotopy theory.

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