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# Algebraic and Geometric Connecting Homomorphisms in the Adams Spectral Sequence

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Let $E$ be a commutative ring spectrum such that $E_*E$ is flat over $\pi_*E$ and such that, for any spectra $X$ and $Y$, $[X, Y \wedge E] \cong \mathrm{Hom}_{E_*E}(E_*X, E_*Y \otimes_{\pi_*E} E_*E)$ (see, e.g., [1, §13 and §16]).

If $A \to B \to C$ is a cofiber sequence such that (1) is short exact

$$(1) \qquad 0 \to E_*A \to E_*B \to E_*C \to 0$$

then there is an algebraically defined connecting homomorphism

$$\partial: \mathrm{Ext}^{s,t}_{E_*E}(M, E_*C) \to \mathrm{Ext}^{s+1,t}_{E_*E}(M, E_*A)$$

for any $E_*E$ comodule $M$. When $M = E_*X$, these Ext groups are $E_2$ terms of Adams spectral sequences and we may ask:

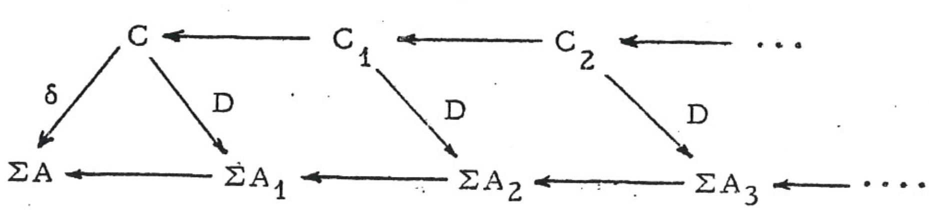
(a) Does $\partial$ commute with differentials in the Adams spectral sequence?

(b) Does $\partial$ converge to the homomorphism $\delta_*: [X, C] \to [X, \Sigma A]$ induced by the geometric connecting map $\delta: C \to \Sigma A$?

It is possible to answer (b) without answering (a) (see [2, Theorem 1.7]). We show here that $\delta$ induces $\partial$ in the most natural possible way, answering (a) and (b) affirmatively.

The canonical Adams resolution of a spectrum $Y$ with respect to $E$ is defined by requiring that $Y_{i+1} \to Y_i \to Y_i \wedge E$ be a cofibration for each $i \geq 0$.

<u>Lemma</u>: The connecting map $\delta: C \to \Sigma A$ induces a map $D$ of Adams resolutions with a shift of filtration:



**Proof.** Since $E_*(\delta) = 0$, our assumptions on E imply that $C \to \Sigma A \to \Sigma A \wedge E$ is nullhomotopic. The existence of D now follows just as in the proof that a map of spectra induces a map of Adams resolutions.

Let $E_r^{**}(X, Y)$ be the $E_r$ term of the Adams spectral sequence for $[X, Y]^E$ and let $F^s[X, Y] = \mathrm{Im}([X, Y_s] \to [X, Y])$ (so that $E_\infty^{s*} = F^s/F^{s+1}$).

By composing with D we obtain a map of exact couples and hence a map of Adams spectral sequences $\{D_r\}: \{E_r^{s,t}(X, C)\} \to \{E_r^{s+1,t}(X, A)\}$. By the lemma, $\delta_* F^s[X, C] \subset F^{s+1}[X, A]$ and therefore the ordinary associated graded homomorphism $E^0(\delta_*): E_\infty^{s*}(X, C) \to E_\infty^{s*}(X, A)$ is zero. Because of the filtration shift, $\delta_*$ induces a homomorphism $E_\infty^{s,t}(X, C) \to E_\infty^{s+1,t}(X, A)$ and this is clearly $D_\infty$, the homomorphism induced by composition with D. It follows that in order to answer (a) and (b) affirmatively we need only show that $D_2$ is the connecting homomorphism for Ext.

**Proposition.** The connecting homomorphism

$$\mathrm{Ext}^{s,t}_{E_*E}(E_*X, E_*C) \to \mathrm{Ext}^{s+1,t}_{E_*E}(E_*X, E_*A)$$

induced by the short exact sequence (1) preserves all differentials and converges to $\delta_*$.

**Proof.** Interpreting Ext as equivalence classes of exact sequences, the connecting homomorphism is Yoneda composite with (1). On the other hand, the homomorphism induced by D is the homomorphism induced by $D_*: E_*C \to E_*\Sigma A_1$ followed by Yoneda composite with $E_*A \to E_*(A \wedge E) \to E_*\Sigma A_1$. This is obvious from the following diagram if one keeps in mind both definitions of Ext:

(i) cocycles modulo coboundaries, (ii) equivalence classes of exact sequences.

$$
\begin{array}{ccccccccccccc}
 & & & & 0 & \to & E_*C & \to & E_*(C \wedge E) & \to & E_*(\Sigma C_1 \wedge E) & \to & \cdots \\
 & & & & & & \downarrow D_* & & \downarrow D_* & & \downarrow D_* & & \\
0 & \to & E_*A & \to & E_*(A \wedge E) & \to & E_*\Sigma A_1 & \to & E_*(\Sigma A_1 \wedge E) & \to & E_*(\Sigma^2 A_2 \wedge E) & \to & \cdots \\
 & & & & & \nearrow & & \searrow & & & & & \\
 & & & & 0 & & & & 0 & & & &
\end{array}
$$

Thus we need only show that there exists a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & E_*A & \to & E_*B & \to & E_*C & \to & 0 \\
 & & \| & & \downarrow & & \downarrow D_* & & \\
0 & \to & E_*A & \to & E_*(A \wedge E) & \to & E_*\Sigma A_1 & \to & 0
\end{array}
$$

The existence of such a diagram follows immediately from the map of cofiber sequences induced by D

$$
\begin{array}{ccccccc}
A & \to & B & \to & C & \xrightarrow{\delta} & \Sigma A \\
\| & & \downarrow & & \downarrow D & & \| \\
A & \to & A \wedge E & \to & \Sigma A_1 & \to & \Sigma A
\end{array}
$$


[1] J.F. Adams. Stable Homotopy and Generalized Homology. Univ. Chicago Lect. Notes in Math. 1974

[2] Johnson, Miller, Wilson, Zahler. Boundary Homomorphisms in the Generalized Adams Spectral Sequence and the Nontriviality of Infinitely Many $\gamma_t$ in Stable Homotopy. Proc. of the Conf. on Homotopy Theory, Northwestern Univ., 1974, Notas de Matematica y Simposia, Sociedad Matematica Mexicana.