Characteristic Classes in K-Theory
Connective K-theory of BG

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Outline

All the representation rings we consider, except those of tori and 2-tori, come with a defining representation whose exterior powers generate the representation ring (except for $SO(2n)$ which requires one additional generator).

We will denote these by $\lambda_i(k^n)$, with $k = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ if it is necessary to be completely precise, abbreviating this to λ_i^k or simply λ_i when possible without ambiguity. In particular, the defining representation is λ_1.

$$RU(T^n) = \mathbf{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

where t_i is the one dimensional representation obtained by projecting onto the i-th factor. All simple representations but the trivial one are complex. The integral cohomology ring is

$$H^* BT^n = \mathbf{Z}[y_1, \ldots, y_n]$$

with $y_i = c_1(t_i)$.

Theorem

$$ku^* BT^n = ku^*[[y_1, \ldots, y_n]]$$

with $y_i = c_1^{ku}(t_i)$ *and*

$$ku^*_{T^n} = \mathrm{MRees}(RU(T^n)) = ku^*[y_1, \overline{y}_1 \ldots, y_n, \overline{y}_n]/(vy_i\overline{y}_i = y_i + \overline{y}_i)$$

with $\overline{y}_i = c_1^{ku}(t_i^{-1})$.

$$RU(Sp(n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_n]$$

The λ_{2i} are real and the λ_{2i+1} are quaternionic. In particular, they are all self conjugate. Note that $\lambda_1 = \mathbf{H}^n = \mathbf{C}^{2n}$, which is $2n$ dimensional, but its higher exterior powers $\lambda_{n+1}, \ldots \lambda_{2n}$ can be expressed in terms of the first n.

The integral cohomology is

$$H^* BSp(n) = \mathbf{Z}[p_1, \ldots, p_n]$$

with $|p_i| = 4i$.

Restriction along $Sp(1)^n \longrightarrow Sp(n)$ will play much the same role for $Sp(n)$ as restriction along $T(n) = U(1)^n \longrightarrow U(n)$ plays for $U(n)$, so we start by considering $Sp(1)^n$.

$$RU(Sp(1)^n) = \mathbf{Z}[s_1, \ldots, s_n]$$

where s_i is the one dimensional symplectic representation obtained by projecting onto the i-th factor. Complexification factors as $Sp(1) \cong SU(2) \subset U(2)$ so we have $c_1^{ku}(s_i) = vc_2^{ku}(s_i)$ and

$$v^2 c_2^{ku}(s_i) = vc_1^{ku}(s_i) = c_2^R(s_i) = c_1^R(s_i) = 2 - s_i.$$

Thus, we have classes $z_i = c_2^{ku}(s_i) \in ku^4(BSp(1)^n)$ which satisfy

$$v^2 z_i = 2 - s_i.$$

We will see that z_i comes from ko^*. The integral cohomology ring is

$$H^* BSp(1)^n = \mathbf{Z}[z_1, \ldots, z_n]$$

with $z_i = p_1(s_i)$, the first Pontrjagin class of s_i.

Theorem

There are compatible generators z_i so that

- $ku^* BSp(1)^n = ku^*[[z_1, \ldots, z_n]]$
- $ko^* BSp(1)^n = ko^*[[z_1, \ldots, z_n]]$
- $ku^*_{Sp(1)^n} = ku^*[z_1, \ldots, z_n] = \mathrm{MRees}(RU(Sp(1)^n))$
- $ko^*_{Sp(1)^n} = ko^*[z_1, \ldots, z_n]$

In particular, $z_i^{ku} \in ku^4_{Sp(1)^n}$ and $z_i^{ko} \in ko^4_{Sp(1)^n}$ satisfy

- $v^2 z_i^{ku} = 2 - s_i \in ku^0_{Sp(1)^n} = RU(Sp(1)^n)$,
- $\alpha z_i^{ko} = 2(2 - s_i) \in ko^0_{Sp(1)^n} = RO(Sp(1)^n)$ *and*
- $\beta z_i^{ko} = 2 - s_i \in ko^{-4}_{Sp(1)^n} = RSp(Sp(1)^n)$.

Proof.

The Adams spectral sequence collapses at

$$E_2^{*,*} = H^*BSp(1)^n \otimes \mathrm{Ext}^{*,*}_{\mathcal{A}(1)}(\mathbf{F}_2, \mathbf{F}_2) \Longrightarrow ko^*BSP(1)^n$$

and similarly for $E(1)$ and ku^*.

The equivariant cases then follow by the defining pullback squares

$$\begin{tikzcd} ku^*_{Sp(1)^n} \arrow[r] \arrow[d] & KU^*[z_1, \ldots, z_n] \arrow[d] \\ ku^*[[z_1, \ldots, z_n]] \arrow[r] & KU^*[[z_1, \ldots, z_n]] \end{tikzcd}$$

The periodic groups are as claimed because we can change generators from the s_i to the $z_i = (2 - s_i)/v^2$. This is $\mathrm{MRees}(RU(Sp(1)^n))$: all irreducible representations are two dimensional, so $JU_{2n} = JU_{2n-1} = (JU_2)^n$. $\square$

Representation theoretic description

Write $z^{\langle i\rangle} = z_1^{i_1} \cdots z_n^{i_n}$ for any monomial in the $z_1, \ldots, z_n$ with exponent sum $i_1 + \cdots + i_n = i$. Then

$$\begin{aligned}
RO &= \mathbf{Z}\langle z^{\langle 2i\rangle}, 2z^{\langle 2i+1\rangle} \mid i \geq 0\rangle \\
RSp &= \mathbf{Z}\langle 2z^{\langle 2i\rangle}, z^{\langle 2i+1\rangle} \mid i \geq 0\rangle \\
JSp_{2k} &= \mathbf{Z}\langle z^{\langle 2i\rangle}, 2z^{\langle 2i+1\rangle} \mid i \geq k\rangle \\
JSp_{2k-1} &= \mathbf{Z}\langle z^{\langle 2i-1\rangle}, 2z^{\langle 2i\rangle} \mid i \geq k\rangle \\
JO \cdot JSp_{2k} &= \mathbf{Z}\langle 2z^{\langle 2i+1\rangle}, z^{\langle 2i+2\rangle} \mid i \geq k\rangle \\
JO \cdot JSp_{2k-1} &= \mathbf{Z}\langle 2z^{\langle 2i\rangle}, z^{\langle 2i+1\rangle} \mid i \geq k\rangle
\end{aligned}$$

and

$$JU_{2k} = JU_{2k-1} = JU^k = \mathbf{Z}\langle z^{\langle i\rangle} \mid i \geq k\rangle$$

Then

$$\frac{RU}{RO} \supset \frac{JU_{4k-4}}{JSp_{2k-2}} = \frac{JU_{4k-2}}{JO \cdot JSp_{2k-2}} = \mathbf{F}_2\langle z^{\langle 2i-1\rangle} \mid i \geq k\rangle$$

$$\frac{RSp}{RU} \supset \frac{JSp_{2k-1}}{JU_{4k-2}} = \frac{JO \cdot JSp_{2k-3}}{JU_{4k-4}} = \mathbf{F}_2\langle z^{\langle 2i-1\rangle} \mid i \geq k\rangle$$

$$\frac{RU}{RSp} \supset \frac{JU_{4k-2}}{JSp_{2k-1}} = \frac{JU_{4k}}{JO \cdot JSp_{2k-1}} = \mathbf{F}_2\langle z^{\langle 2i\rangle} \mid i \geq k\rangle$$

$$\frac{RO}{RU} \supset \frac{JSp_{2k}}{JU_{4k}} = \frac{JO \cdot JSp_{2k-2}}{JU_{4k-2}} = \mathbf{F}_2\langle z^{\langle 2i\rangle} \mid i \geq k\rangle$$

Corollary

The coefficients of $Sp(1)^n$-equivariant connective real K-theory are

$$ko_i^{Sp(1)^n} = \begin{cases} 0 & i = -8k+7 \\ JU_{4k-3}/JO \cdot JSp_{2k-2} \subset RU/RO & i = -8k+6 \\ JSp_{2k-1}/JU_{4k-3} \subset RSp/RU & i = -8k+5 \\ JSp_{2k-1} \subset RSp & i = -8k+4 \\ 0 & i = -8k+3 \\ JU_{4k-2}/JSp_{2k-1} \subset RU/RSp & i = -8k+2 \\ JSp_{2k}/JU_{4k} \subset RO/RU & i = -8k+1 \\ JSp_{2k} \subset RO & i = -8k \end{cases}$$

Proof.

Collapse of the Adams spectral sequence implies each $z^{\langle i \rangle} = z_1^{i_1} \cdots z_n^{i_n}$ contributes a copy of ko_* shifted to start in $ko_{-4i}^{Sp(1)^n}$. Now, $\alpha z^{\langle i \rangle} \in ko_{-4i+4}^{Sp(1)^n}$ will map to $2z^{\langle i \rangle} \in RU(Sp(1)^n)$ since $\alpha \mapsto 2v^2$. Similar factors of 2 appear when mapping RU to RO or RSp here. Thus

$$
\begin{array}{lcl}
ko_{-8k+6} & = & \mathbf{F}_2\langle z^{\langle 2i-1 \rangle} \mid i \geq k \rangle \\
ko_{-8k+5} & = & \mathbf{F}_2\langle z^{\langle 2i-1 \rangle} \mid i \geq k \rangle \\
ko_{-8k+4} & = & \mathbf{Z}\langle z^{\langle 2i-1 \rangle}, 2z^{\langle 2i \rangle} \mid i \geq k \rangle \\
ko_{-8k+2} & = & \mathbf{F}_2\langle z^{\langle 2i \rangle} \mid i \geq k \rangle \\
ko_{-8k+1} & = & \mathbf{F}_2\langle z^{\langle 2i \rangle} \mid i \geq k \rangle \\
ko_{-8k} & = & \mathbf{Z}\langle z^{\langle 2i \rangle}, 2z^{\langle 2i+1 \rangle} \mid i \geq k \rangle
\end{array}
$$

Multiplication by β^k embeds these into the periodic part of $ko_*^{Sp(1)^n}$, determining, for example, that we will write $ko_{-8k+6}^{Sp(1)^n}$ as the quotient of something contained in RU by something in RSp. □

Pontrjagin classes

Definition

The k^{th} representation theoretic *Pontrjagin class* of an n-dimensional symplectic representation $V : G \longrightarrow Sp(n)$ is

$$p_k^R(V) = \sum_{j=0}^{k} (-1)^j 2^{k-j} \binom{n-j}{n-k} \Lambda^j(V)$$

Proposition

The restriction $RU(Sp(n)) \longrightarrow RU(Sp(1)^n)$ sends p_k^R to $\sigma_k(2-s_1,\ldots,2-s_n)$. The representation p_k^R is real if k is even, and quaternionic if k is odd.

Accordingly, we shall generally consider p_{2i}^R as an element of $RO(G)$ and p_{2i+1}^R as an element of $RSp(G)$. Note, however, that representations which are not irreducible can be both real and quaternionic.

Theorem

We have

- $ku^* BSp(n) = ku^*[[p_1, \ldots, p_n]]$
- $ko^* BSp(n) = ko^*[[p_1, \ldots, p_n]]$
- $ku^*_{Sp(n)} = ku^*[p_1, \ldots, p_n]$
- $ko^*_{Sp(n)} = ko^*[p_1, \ldots, p_n]$.

In each case, p_k restricts to $\sigma_k(z_1, \ldots, z_n)$.

In ku^*, $v^{2k} p_k^{ku} = p_k^R \in ku^0_{Sp(n)} = RU(Sp(n))$.

In ko^*, $\beta^k p_{2k}^{ko} = p_{2k}^R \in ko^0_{Sp(n)} = RO(Sp(n))$ and $\beta^k p_{2k+1}^{ko} = p_{2k+1}^R \in ko^4_{Sp(n)} = JSp(Sp(n))$.

Definition

Let $V : G \longrightarrow Sp(n)$ be a symplectic representation. For $E = RU$, ko, KO, ku, KU or H, we define the *Pontrjagin class* $p_i^E(V) \in E_G^{4i}$ to be $V^*(p_i)$. It is convenient to collect these into the *total Pontrjagin class*

$$p_\bullet^E(V) = 1 + p_1^E(V) + p_2^E(V) + \cdots + p_n^E(V)$$

and to let $p_i^E(V) = 0$ if $i > n$.

Corollary

$$p_\bullet^E(V \oplus W) = p_\bullet^E(V)p_\bullet^E(W)$$

Lemma

The restriction $ku^*_{Sp(1)^n} \longrightarrow ku^*_{T(n)}$ *maps* z_i *to* $y_i\overline{y}_i$.

Write $\overline{c}_i(V) = c_i(\overline{V})$ for the Chern classes of the complex conjugate of a representation.

Theorem

The restriction maps $ku^*_{U(2n)} \xrightarrow{q^*} ku^*_{Sp(n)} \xrightarrow{\widetilde{c}^*} ku^*_{U(n)}$ *obey*

$$c_k \mapsto \sum_{0 \le 2i \le k} \binom{k-i}{i} v^{k-2i} p_{k-i} \mapsto \sum_{i+j=k} c_i \overline{c}_j.$$

Specializing to ordinary cohomology by setting $v = 0$ we obtain the usual relations (up to sign):

$$q^*(p_n^H) = \sum_{i+j=2n} c_i^H \overline{c}_j^H = \sum_{i+j=2n} (-1)^j c_i^H c_j^H,$$

$$\widetilde{c}^*(c_{2i-1}) = 0, \qquad \text{and} \qquad \widetilde{c}^*(c_{2i}) = p_i.$$

For $Sp(4)$, for example,

$$\begin{array}{llllllll}
c_1 \mapsto & vp_1 & & & & \mapsto & c_1 + \overline{c}_1 \\
c_2 \mapsto & p_1 & +v^2p_2 & & & \mapsto & c_2 + c_1\overline{c}_1 + \overline{c}_2 \\
c_3 \mapsto & & 2vp_2 & +v^3p_3 & & \mapsto & c_3 + c_2\overline{c}_1 + c_1\overline{c}_2 + \overline{c}_3 \\
c_4 \mapsto & & p_2 & +3v^2p_3 & +v^4p_4 & \mapsto & c_4 + c_3\overline{c}_1 + c_2\overline{c}_2 + c_1\overline{c}_3 + \overline{c}_4 \\
c_5 \mapsto & & & 3vp_3 & +4v^3p_4 & \mapsto & c_4\overline{c}_1 + c_3\overline{c}_2 + c_2\overline{c}_3 + c_1\overline{c}_4 \\
c_6 \mapsto & & & p_3 & +6v^2p_4 & \mapsto & c_4\overline{c}_2 + c_3\overline{c}_3 + c_2\overline{c}_4 \\
c_7 \mapsto & & & & 4vp_4 & \mapsto & c_4\overline{c}_3 + c_3\overline{c}_4 \\
c_8 \mapsto & & & & p_4 & \mapsto & c_4\overline{c}_4
\end{array}$$

It is more difficult to get good expressions for the images of the individual p_i. However, for $i = 1$, using the fact that v acts monomorphically on $ku^*_{U(n)}$ we have

$$p_1 \mapsto \frac{c_1 + \overline{c}_1}{v} = \sum_{k=2}^{n} (-v)^{k-2} \sum_{i+j=k} c_i \overline{c}_j$$
$$= c_2 + c_1\overline{c}_1 + \overline{c}_2 - v \sum_{k=3}^{n} (-v)^{k-3} \sum_{i+j=k} c_i \overline{c}_j.$$

In cohomology, where $v = 0$ and $\overline{c}_i = (-1)^i c_i$, we have

$$p_1 \mapsto c_2 + c_1\overline{c}_1 + \overline{c}_2 = 2c_2 - c_1^2$$

with our normalization of the p_i. Thus, if $c_1 = 0$, then $c_2 = p_1/2$.

Finally, we provide the following *symplectic splitting principle*.

Theorem

Let ξ be an $Sp(n)$ bundle over X. Then there exists a map $f : Y \longrightarrow X$ such that $f^*\xi$ is a sum of symplectic line bundles and $f^* : H^*X \longrightarrow H^*Y$ is a monomorphism.

Proof.

Let Y be the pullback

$$\begin{array}{ccc} Y & \longrightarrow & BSp(1)^n \\ \downarrow{\scriptstyle f} & & \downarrow \\ X & \longrightarrow & BSp(n) \end{array}$$

along the classifying map of the bundle ξ. The universal bundle over $BSp(n)$ splits as a sum of line bundles over $BSp(1)^n$, so $f^*\xi$ also splits in this manner.

The Serre spectral sequence gives the cohomology statement. □

We can compute $ko_* BSp(n)$ by the collapsed local cohomology spectral sequence because the adjoint representation is *ko* oriented, being a *Spin* bundle, since w_1 and w_2 are trivially 0, so that

$$H^n_J(ko^*(BSp(n)) = ko_*(BSp(n)^{ad}) = \Sigma^{\dim(Sp(n))} ko_*(BSp(n)).$$

$$RU(SU(n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_{n-1}].$$

We have $\overline{\lambda_i} = \lambda_{n-i}$, so the λ_i are all complex unless $n = 2m$, when λ_m is real if m is even and quaternionic if m is odd. The integral cohomology is

$$H^* BSU(n) = \mathbf{Z}[c_2, \ldots, c_n]$$

with $c_i = c_i(\lambda_1)$. The connective complex K-theory is easy to compute.

Theorem

$ku^* BSU(n) = ku^*[[c_2, \ldots, c_n]]$ and

$$ku^*_{SU(n)} = \mathrm{MRees}(RU(SU(n))) = ku^*[c_2, \ldots, c_n].$$

Proof.

Since $H^*BSU(n)$ is concentrated in even degrees, the Atiyah-Hirzebruch spectral sequence implies $ku^*BSU(n)$ must be the complete ku^* algebra freely generated by $c_2, \ldots, c_n$.

In $KU^*_{SU(n)}$, we have

$$\lambda_i = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} c_j^R = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} v^j c_j^{ku}.$$

and $\lambda_n = 1$.

Hence, the $c_j = c_j^{ku}$ generate, and $c_1 - vc_2 + \cdots + (-v)^{n-1} c_n = 0$. Thus $KU^*_{SU(n)}$ is polynomial on any $n-1$ of $c_1, \ldots, c_n$. In particular, $KU^*_{SU(n)} = KU^*[c_2, \ldots, c_n]$. □

Proof.

(Cont.) The pullback square

$$\begin{tikzcd}
ku^*_{SU(n)} \arrow[r] \arrow[d] & KU^*[c_2, \ldots, c_n] \arrow[d] \\
ku^*[[c_2, \ldots, c_n]] \arrow[r] & KU^*[[c_2, \ldots, c_n]]
\end{tikzcd}$$

shows that $ku^*_{SU(n)} = ku^*[c_2, \ldots, c_n]$. □

$$RU(U(n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}]$$

The integral cohomology is

$$H^* BU(n) = \mathbf{Z}[c_1, \ldots, c_n]$$

where $c_i = c_i(\lambda_1)$. Again, the complex connective K-theory follows immediately.

Theorem

$ku^* BU(n) = ku^*[[c_1, \ldots, c_n]]$ and

$$ku^*_{U(n)} = \mathrm{MRees}(RU(U(n))) = ku^*[c_1, \ldots, c_n, \Delta^{-1}]$$

where $\Delta = \lambda_n = 1 - vc_1 + v^2 c_2 - \cdots + (-v)^n c_n$.

Proof.

The argument is nearly the same as for $SU(n)$, except that $KU^*_{U(n)}$ is not polynomial, but is instead $KU^*[c_1, \ldots, c_n, \Delta^{-1}]$. $\square$

In cohomology, restriction along the inclusion $SU(n) \longrightarrow U(n)$ is the quotient which sends c_1 to 0. The proper way to think of this is that it is taking the quotient by the Chern class of the determinant of the defining representation. In K-theory, the map this induces is more interesting.

Theorem

The restriction homomorphism $ku^*_{U(n)} \longrightarrow ku^*_{SU(n)}$ is the quotient $ku^*[c_1, \ldots, c_n, \Delta^{-1}] \longrightarrow ku^*[c_2, \ldots, c_n]$ which sends Δ to 1 and c_1 to $vc_2 - v^2c_3 + \cdots - (-v)^{n-1}c_n$.

Proof.

$SU(n)$ is the kernel of the determinant $U(n) \longrightarrow U(1)$. The determinant sends $y = (1 - \lambda_1)/v \in ku^2_{U(1)}$ to $(1 - \lambda_n)/v = c_1 - vc_2 + v^2c_3 - \cdots$, so this must go to zero in $ku^*_{SU(n)}$. After dividing by this, we have an isomorphism, by the calculation of $ku^*_{SU(n)}$. □

Consider the conjugate Chern classes $\overline{c}_i(V) = c_i(\overline{V})$.

Corollary

The restriction homomorphism $ku^*_{U(n)} \longrightarrow ku^*_{T(n)}$ *sends* $\overline{c}_i$ *to* $\sigma_i(\overline{y}_1, \ldots, \overline{y}_n)$. *The* $\overline{c}_i$ *also satisfy*

- $\Delta v^i \overline{c}_i = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} \lambda_{n-j}$
- $\Delta \overline{c}_i = \sum_{k=i}^{n} (-1)^k \binom{k}{i} v^{k-i} c_k$

The conjugate $\overline{\Delta} = \overline{\lambda}_n = 1 - v\overline{c}_1 + v^2\overline{c}_2 - \cdots \pm v^n\overline{c}_n$ satisfies $\Delta\overline{\Delta} = 1$. Collecting terms we find

Proposition

In $ku^*_{U(n)}$,

$$c_1 + \overline{c}_1 = -\sum_{k=2}^{2n} (-v)^{k-1} \sum_{i+j=k} c_i \overline{c}_j$$

$$RU(O(n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_n]/(\lambda_n^2 - 1, \lambda_i\lambda_n - \lambda_{n-i}).$$

These representations are all real, so that complexification and quaternionification are isomorphisms

$$RO(O(n)) \xrightarrow{\cong} RU(O(n)) \xrightarrow{\cong} RSp(O(n)).$$

The integral cohomology is complicated. The best approach is to give the mod 2 cohomology, and if integral issues matter, the cohomology localized away from 2. We have

$$H\mathbf{F}_2^* BO(n) = \mathbf{F}_2[w_1, \ldots, w_n]$$

where $w_i = w_i(\lambda_1)$.

Rewrite the representation ring in terms of Chern classes, as usual: let $c_i = c_i^K(\lambda_1) \in K^{2i}_{O(n)}$, so that

$$\lambda_i = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} v^j c_j.$$

Rather than replace λ_n by the top Chern class, c_n, we use the first Chern class of the determinant representation, $c = c_1^K(\lambda_n) \in K^2_{O(n)}$. This satisfies $vc = 1 - \lambda_n$, which is much more convenient than $v^n c_n = 1 - \lambda_1 + \cdots + (-1)^n \lambda_n$.

Proposition

$$KU^*_{O(2n+1)} = KU^*[c_1, \ldots, c_n, c]/(vc^2 - 2c)$$

and

$$KU^*_{O(2n)} = KU^*[c_1, \ldots, c_n, c]/(vc^2 - 2c, c \sum_{i=0}^{n} \binom{2n-i}{n} (-v)^i c_i)$$

To compute $ku^*BO(n)$, we need to determine the $E(1)$-module structure of $H\mathbf{F}_2^*BO(n)$. We start with its stable type. Let ϵ be 0 or 1.
First, the submodule

$$\mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n}^2] \longrightarrow H^*BO(2n+\epsilon)$$

is a trivial $E(1)$-submodule.
Second, the reduced homology of $BO(1)$ is the ideal (w_1) in $\mathbf{F}_2[w_1]$, and as an $E(1)$-submodule,

$$(w_1) \otimes \mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n-2}^2] \longrightarrow H^*BO(2n-\epsilon)$$

is a direct sums of suspensions of (w_1).
The sum of these two submodules exhausts the 'interesting' part of $H^*BO(n)$, in the sense that the complementary summand is $E(1)$-free.

Theorem

The inclusions

$$\mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n}^2] \oplus (w_1) \otimes \mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n-2}^2] \longrightarrow H^*BO(2n)$$

and

$$\mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n}^2] \oplus (w_1) \otimes \mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n}^2] \longrightarrow H^*BO(2n+1)$$

induce isomorphisms in Q_0 and Q_1 homology.

Corollary

As an $E[Q_0, Q_1]$-module, $H^*BO(n)$ is the sum of trivial modules, suspensions of $H^*BO(1)$, and free modules.

Proof.

The corollary follows by the result of Adams and Margolis, that Q_0 and Q_1 homology detects the stable isomorphism type of the module. $\square$

In principle, this describes $H^*BO(n)$ as an $E(1)$-module but finding a good parametrization of the complementary $E(1)$-free submodule is non-trivial.
The $\mathcal{A}(1)$-module structure is not as simple, as Sq^2 does not annihilate all squares.
The w_{2i}^2 detect Pontrjagin classes p_i of the defining representation and w_1^2 in the (w_1) summand detects the first Chern class of the determinant representation.

Corollary

The Adams spectral sequence converging to $ku^*BO(n)$ collapses at E_2, and the natural homomorphism

$$ku^*BO(n) \longrightarrow H^*BO(n) \oplus KU^*BO(n)$$

is a monomorphism.

Comments on the proof

The $H(-, Q_0)$ isomorphism is straightforward, but the $H(-, Q_1)$ isomorphism requires a careful choice of generators.

Once the correct generators are identified, it turns out that the general case is just a regraded version of $H^*BO(4)$ tensored with an $E(1)$-trivial subalgebra.

See the book with Greenlees for details.

$O(1)$

Recall that $ku^*_{O(1)} = ku^*[c]/(vc^2 - 2c)$ by the pullback square

$$
\begin{tikzcd}
ku^*_{O(1)} \arrow[r] \arrow[d] & KU^*_{O(1)} = KU^*[c]/(vc^2 - 2c) \arrow[d] \\
ku^*BO(1) = ku^*[[c]]/(vc^2 - 2c) \arrow[r] & KU^*BO(1) = KU^*[[c]]/(vc^2 - 2c)
\end{tikzcd}
$$

The Bockstein spectral sequence then gives

Theorem

There are unique elements $p_0 \in ko^0_{O(1)}$ and $p_1 \in ko^4_{O(1)}$ with complexifications $c(p_0) = vc$ and $c(p_1) = c^2$. The ring

$$
ko^*_{O(1)} = \frac{ko^*[p_0, p_1]}{(\eta p_1, \alpha p_1 - 4p_0, \beta p_1 - \alpha p_0, p_0 p_1 - 2p_1, p_0^2 - 2p_0)}
$$

In terms of representation theory, this can be written as follows.

Corollary

$O(1)$-equivariant connective real K-theory has coefficient ring

$$ko^i_{O(1)} = \begin{cases} RSp & i = -8k-4 \leq 0 \\ RO/2 & i = -8k-2 \leq 0 \\ RO/2 & i = -8k-1 \leq 0 \\ RO & i = -8k \leq 0 \\ JSp_k = JSp^k & i = 4k > 0 \\ 0 & \text{otherwise} \end{cases}$$

To justify the notation p_i:

Theorem

The restriction $ku^*_{Sp(1)} \longrightarrow ku^*_{O(1)}$ is:

$$z = p_1(\lambda_1) \mapsto p_1, \qquad \alpha z \mapsto 4p_0, \qquad \text{and} \qquad \beta z \mapsto \alpha p_0,$$

Proof.

That z maps to p_1 is evident by comparison with ku^*. The rest follows by the relations in $ko^*_{O(1)}$. □

Thus, p_1 really is the first Pontrjagin class of the quaternionic representation induced up from the defining representation of $O(1)$, while p_0 is a genuinely real class. We call it p_0 because of the relations which tie it to p_1.

$O(2)$

Corollary

$$\begin{aligned} KU^*_{O(2)} &= KU^*[c, c_1]/(vc^2 - 2c, c(2 - vc_1)) \\ &= KU^*[c, c_2]/(vc^2 - 2c, cc_2) \end{aligned}$$

Proof.

The calculation

$$\begin{aligned} v^2c_2 &= 1 - \lambda_1 + \lambda_2 \\ &= 1 - (2 - vc_1) + (1 - vc) \\ &= v(c_1 - c) \end{aligned}$$

shows that $c_1 = c + vc_2$. Then the relation $0 = c(2 - vc_1)$ becomes $v^2cc_2 = 0$ since $c(2 - vc) = 0$. $\square$

The connective K-theory is similar but somewhat larger.

Theorem

$$ku^*_{O(2)} = ku^*[c, c_2]/(vc^2 - 2c, 2cc_2, vcc_2)$$

Proof.

Decomposing $H^*BO(2)$ as an $E[Q_0, Q_1]$-module shows that c and c_2 are algebra generators for $ku^*BO(2)$. The monomorphism into $H^*BO(2) \oplus KU^*BO(2)$ then shows the relations are complete. The pullback square then gives us $ku^*_{O(2)}$. □

The Bockstein spectral sequence then gives

Theorem

$KO^*_{O(2)} = KO^*[p_0, r_0]/(p_0^2 - 2p_0, p_0 r_0)$ *and*

$$ko^*_{O(2)} = ko^*[p_0, p_1, p_2, r_0, r_1, s]/I$$

where I is the ideal generated by the relations

$$\begin{array}{lll} \eta p_1 = 0 & \alpha p_1 = 4p_0 & \beta p_1 = \alpha p_0 \\ \eta r_1 = 0 & \alpha r_1 = 4r_0 & \beta r_1 = \alpha r_0 \\ \eta s = 0 & \alpha s = 0 & \beta s = \eta^2 r_0 \\ \hline p_0 p_2 = 0 & p_1 p_2 = s^2 & \end{array}$$

	p_0	p_1	r_0	r_1	s
p_0	$2p_0$	$2p_1$	0	0	0
p_1	$2p_1$	p_1^2	0	0	$p_1 s$
r_0	0	0	βp_2	αp_2	$\eta^2 p_2$
r_1	0	0	αp_2	$4p_2$	0

p_0 and r_0 are $1 - \det$ and the Euler class of the defining representation, respectively.

p_1 and r_1 are their images in $JSp = ko^4$. This explains the similarity of the action of ko^* on them.

The class p_2 refines the square of the Euler class in the sense that $r_1^2 = 4p_2$, $r_0 r_1 = \alpha p_2$ and $r_0^2 = \beta p_2$.

The class s is a square root of the product $p_1 p_2 = s^2$.

The relation $\beta s = \eta^2 r_0$ is hidden in the Bockstein spectral sequence. Representation theory (i.e., the map into $KO^*_{O(2)}$) and the Adams spectral sequence each work to recover it.

The images of the natural maps from $ko^*_{O(2)}$ to $ku^*_{O(2)}$, $KO^*_{O(2)}$ and $H\mathbf{F}_2^* BO(2)$ are as follows:

$ko^*_{O(2)}$	$ku^*_{O(2)}$	$KO^*_{O(2)}$	$H^* BO(2)$
p_0	vc	p_0	0
p_1	c^2	$\alpha\beta^{-1} p_0$	w_1^4
p_2	c_2^2	$\beta^{-1} r_0^2$	w_2^4
r_0	$v^2 c_2$	r_0	0
r_1	$2c_2$	$\alpha\beta^{-1} r_0$	0
s	cc_2	$\eta^2\beta^{-1} r_0$	$w_1^2 w_2^2$

$O(3)$

Corollary

$KU^*_{O(3)} = KU^*[c, c_1]/(vc^2 - 2c)$

Proposition

$KO^*_{O(3)} = KO^*[p_0, q_0]/(p_0^2 - 2p_0)$ *where* p_0 *and* q_0 *have complexifications* vc *and* vc_1 *respectively. The restriction* $KO^*_{O(3)} \longrightarrow KO^*_{O(2)}$ *sends* p_0 *to* p_0 *and* q_0 *to* $p_0 + r_0$.

The Chern classes no longer suffice to generate $ku^*_{O(n)}$ for $n > 2$. Let $\overline{Q}_0 : H \longrightarrow \Sigma H\mathbf{Z}$ and $\overline{Q}_1 : H\mathbf{Z} \longrightarrow \Sigma^3 ku$ be the boundary maps in the cofiber sequences for $2 : H\mathbf{Z} \longrightarrow H\mathbf{Z}$ and $v : \Sigma^2 ku \longrightarrow ku$. They are lifts of the Milnor primitives Q_0 qnd Q_1.

Definition

Let $q_2 = \overline{Q}_1\overline{Q}_0(w_2) \in ku^6BO(3)$ and $q_3 = \overline{Q}_1\overline{Q}_0(w_3) \in ku^7BO(3)$.

Proposition

The classes q_2 and q_3 are nonzero classes annihilated by $(2, v)$. The class q_3 is independent of c, c_2, and c_3, while $q_2 = cc_2 - 3c_3$. These are the only nonzero 2 or v-torsion classes in $ku^6BO(3)$ and $ku^7BO(3)$.

Theorem

$ku^*BO(3) = ku^*[c, c_2, c_3, q_3]/R$, where R is an ideal containing $(vc^2 - 2c, 2(cc_2 - 3c_3), v(cc_2 - 3c_3), 2q_3, vq_3, vcc_3 - 2c_3)$.

$O(n)$ for larger n

The free summands in $H^*BO(n)$ begin to get more complicated at $n = 4$. Let us write w_S for the product $\prod_{i \in S} w_i$.

Proposition

Maximal $E(1)$-free summands of $H^*BO(n)$ are:

$n = 4$

$$\mathbf{F}_2[w_1^2, w_2^2, w_3^2, w_4^2]\langle w_2, w_3, w_4, w_{234}\rangle$$
$$\oplus \mathbf{F}_2[w_1^2, w_2^2, w_4^2]\langle w_{24}\rangle$$

$n = 5$

$$\mathbf{F}_2[w_1^2, w_2^2, w_3^2, w_4^2, w_5^2]\langle w_2, w_3, w_4, w_5, w_{234}, w_{235}, w_{245}, w_{345}\rangle$$
$$\oplus \mathbf{F}_2[w_1^2, w_2^2, w_4^2, w_5^2]\langle w_{24}, w_{34}\rangle$$

$n = 6$

$$\mathbf{F}_2[w_1^2, w_2^2, w_3^2, w_4^2, w_5^2, w_6^2]\langle w_2, w_3, w_4, w_5, w_6, w_{26}, w_{234}, w_{235}, w$$
$$\oplus \mathbf{F}_2[w_1^2, w_2^2, w_3^2, w_4^2, w_5^2, w_6^2]\langle w_{236}, w_{246}, w_{256}, w_{346}, w_{456}, w_{23456}$$
$$\oplus \mathbf{F}_2[w_1^2, w_2^2, w_4^2, w_5^2, w_6^2]\langle w_{24}, w_{34}, w_{2456}\rangle$$
$$\oplus \mathbf{F}_2[w_1^2, w_2^2, w_4^2, w_6^2]\langle w_{46}\rangle$$

Remark

Each w_S generating a free $E(1)$ will give rise to a $(2, v)$-annihilated class $\overline{Q}_1 \overline{Q}_0(w_S) \in ku^* BO(n)$.

$$RU(SO(2n+1)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_n]$$

with $\lambda_{n+i} = \lambda_{n+1-i}$ and

$$RU(SO(2n)) = \mathbf{Z}[\lambda_1, \ldots, \lambda_{n-1}, \lambda_n^+, \lambda_n^-]/R$$

with $\lambda_{n+i} = \lambda_{n-i}$ and $\lambda_n = \lambda_n^+ + \lambda_n^-$. The ideal R is generated by one relation

$$(\lambda_n^+ + \sum_i \lambda_{n-2i})(\lambda_n^- + \sum_i \lambda_{n-2i}) = (\sum \lambda_{n-1-2i})^2$$

All the λ_i are real. $RU(SO(2n))$ is free over $RU(SO(2n+1))$ on $\{1, \lambda_n^+\}$.

$H^*BSO(n) = \mathbf{F}_2[w_2, \ldots, w_n]$ where $w_i = w_i(\lambda_1)$.
We have already examined $SO(2) = T(1)$ and found (writing c_1 rather than y_1 here)

$$ku^*_{SO(2)} = ku^*[c_1, \overline{c}_1]/(vc_1\overline{c}_1 = c_1 + \overline{c}_1)$$

and

$$ku^*BSO(2) = ku^*[[c_1]].$$

The maps induced in ku^* by the fibre sequence $SO(2) \xrightarrow{i} O(2) \xrightarrow{\det} O(1)$ are

Proposition

$\det^*(c) = c$, *while* $i^*(c) = 0$, $i^*(c_2) = c_1\overline{c}_1$ *and*
$i^*(c_1) = i^*(c + vc_2) = c_1 + \overline{c}_1$.

$SO(3)$

$RU(SO(3)) = \mathbf{Z}[\lambda_1]$ with $\lambda_2 = \lambda_1$ and $\lambda_3 = 1$.

Proposition

$KU^*_{SO(3)} = KU^*[c_2]$

Proof.

The Chern classes of the defining representation of $SO(3)$ satisfy $c_1 = vc_2$, $c_3 = 0$ and $v^2 c_2 = vc_1 = 3 - \lambda_1$. $\square$

Theorem

$$ku^*_{SO(3)} = ku^*[c_2, c_3]/(2c_3, vc_3).$$

The first Chern class, $c_1 = vc_2$. The restriction $ku^*_{O(3)} \longrightarrow ku^*_{SO(3)}$ sends c and q_3 to 0, and sends each c_i to c_i.

Proof.

The Adams spectral sequence again collapses and gives us a monomorphism into the sum of mod 2 cohomology and periodic K-theory. This makes it easy to show $ku^*BSO(3) = ku^*[[c_2, c_3]]/(2c_3, vc_3)$. The pullback square then gives $ku^*_{SO(3)} = ku^*[c_2, c_3]/(2c_3, vc_3)$. $\square$

In general, we expect $c_1 = vc_2 - v^2c_3$, since this is true in $SU(n)$, but here $v^2c_3 = 0$.

The restriction from $O(3)$ is computed by using the monomorphism to periodic K-theory plus mod 2 cohomology. Note that $q_2 = cc_2 - 3c_3$ restricts to c_3.

Thank you