Characteristic Classes in K-Theory
General Theory

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Outline

Representation Rings

Restriction, induction and conjugation induce natural transformations between the real, complex, and quaternionic representation rings:

$$
\begin{array}{ccc}
 & & \mathrm{RO} \\
 & & c \downarrow \;\; \uparrow r \\
\mathrm{RU} & \xleftarrow{\ \tau\ } & \mathrm{RU} \\
 & & \tilde{c} \uparrow \;\; \downarrow q \\
 & & \mathrm{RSp}
\end{array}
$$

$$rc = 2$$

$$cr = 1 + \tau \qquad r\tau = r$$

$$\tau c = c \qquad \tau\tilde{c} = \tilde{c}$$

$$\tilde{c}q = 1 + \tau \qquad q\tau = q$$

$$q\tilde{c} = 2$$

Representation Rings

For any compact Lie group, we may choose

- irreducible real representations U_i,
- irreducible complex representations V_j, and
- irreducible quaternionic representations W_k

so that

- $RU = \mathbf{Z}\langle cU_i \rangle \oplus \mathbf{Z}\langle V_j, \tau V_j \rangle \oplus \mathbf{Z}\langle \tilde{c} W_k \rangle$
- $RO = \mathbf{Z}\langle U_i \rangle \oplus \mathbf{Z}\langle rV_j \rangle \oplus \mathbf{Z}\langle rqW_k \rangle$
- $RSp = \mathbf{Z}\langle qcU_i \rangle \oplus \mathbf{Z}\langle qV_j \rangle \oplus \mathbf{Z}\langle W_k \rangle$

Equivariant K-theory

Evidently,

$$KU^0_G = RU(G)$$

and similarly for KO and KSp. The Atiyah-Segal Theorem asserts that the map $S \longleftarrow EG_+$ induces completion at the augmentation ideal:

$$KU^0_G = RU(G) \longrightarrow KU^0_G(EG_+) = KU^0(BG)$$

and similarly for KO and KSp. Thus

$$KU^0(BG) = RU(G)^{\widehat{}}_I \qquad KO^0(BG) = RO(G)^{\widehat{}}_I \qquad KSp^0(BG) = RSp(G)^{\widehat{}}_I$$

They also show that $[BG, U] = 0 = [BG, O] = [BG, Sp]$.
Write $\widehat{RO}$ for $RO^{\widehat{}}_I$ hereafter.

Using representations of G on Clifford modules, Atiyah, Bott and Shapiro give an elegant account of the Atiyah-Segal isomorphisms, showing:

Theorem

1. $KU_G^* = RU(G)[v, v^{-1}]$.
2. $KO_G^* = RO^*(G)[\beta, \beta^{-1}]$ *where*

$$
\begin{array}{rcll}
RO^0(G) & = & RO(G) & \cong \mathbf{Z}\{U_i, rV_j, r\tilde{c}W_k\} \\
RO^{-1}(G) & = & RO(G)/RU(G) & \cong \mathbf{F}_2\{U_i\} \\
RO^{-2}(G) & = & RU(G)/RSp(G) & \cong \mathbf{F}_2\{cU_i\} \oplus \mathbf{Z}\{\overline{V}_j\} \\
RO^{-3}(G) & = & 0 & \\
RO^{-4}(G) & = & RSp(G) & \cong \mathbf{Z}\{qcU_i, qV_j, W_k\} \\
RO^{-5}(G) & = & RSp(G)/RU(G) & \cong \mathbf{F}_2\{W_k\} \\
RO^{-6}(G) & = & RU(G)/RO(G) & \cong \mathbf{Z}\{\overline{V}_j\} \oplus \mathbf{F}_2\{\tilde{c}W_k\} \\
RO^{-7}(G) & = & 0 &
\end{array}
$$

Coefficients

The action of the coefficients,

$$KU^* = \mathbf{Z}[v, v^{-1}]$$

and

$$KO^* = \frac{\mathbf{Z}[\eta, \alpha, \beta, \beta^{-1}]}{(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)}$$

with $v \in KU^{-2}$, $\eta \in KO^{-1}$, $\alpha \in KO^{-4}$, and $\beta \in KO^{-8}$, coincide with natural maps in representation theory.

For example, η induces the natural quotients

$$RO \longrightarrow RO/RU \qquad \text{and} \qquad RSp \longrightarrow RSp/RU$$

and the evident inclusions

$$RO/RU \longrightarrow RU/RSp \qquad \text{and} \qquad RSp/RU \longrightarrow RU/RO.$$

On the level of Clifford algebras, multiplication by η is complexification.

Similarly multiplication by α is quaternionification. Precisely, it is

- $qc : RO \longrightarrow RSp$ in degrees 0 mod 8
- $r\tilde{c} : RSp \longrightarrow RO$ in degrees 4 mod 8
- multiplication by 2,

  $$\mathbf{F}_2\{cU_i\} \oplus \mathbf{Z}\{\overline{V}_j\} \longrightarrow \mathbf{Z}\{\overline{V}_j\} \oplus \mathbf{F}_2\{\tilde{c}W_k\}$$

  and

  $$\mathbf{Z}\{\overline{V}_j\} \oplus \mathbf{F}_2\{\tilde{c}W_k\} \longrightarrow \mathbf{F}_2\{cU_i\} \oplus \mathbf{Z}\{\overline{V}_j\}$$

  in degrees 2 mod 4,
- and 0 in odd degrees.

There is a more elementary deduction of the values of the completed theory, just from the spaces involved in Bott periodicity, together with the isomorphisms [Atiyah?]

$$[BG, BO \times \mathbf{Z}] = \widehat{RO}(G), \qquad [BG, BSp \times \mathbf{Z}] = \widehat{RSp}(G),$$

and

$$[BG, BU \times \mathbf{Z}] = \widehat{RU}(G), \qquad [BG, U] = 0.$$

Recall that Bott periodicity says that starting with $BO \times \mathbf{Z}$ and repeatedly taking loops gives

- O
- O/U
- U/Sp
- $BSp \times \mathbf{Z}$
- Sp
- Sp/U
- U/O, and then
- $BO \times \mathbf{Z}$ again.

Theorem

The values of $[BG, -]$ *on the infinite loop spaces above are as follows:*

1. $[BG, O] = \widehat{RO}(G)/\widehat{RU}(G)$
2. $[BG, Sp] = \widehat{RSp}(G)/\widehat{RU}(G)$
3. $[BG, U/Sp] = 0$
4. $[BG, U/O] = 0$
5. $[BG, O/U] = \widehat{RU}(G)/\widehat{RSp}(G)$
6. $[BG, Sp/U] = \widehat{RU}(G)/\widehat{RO}(G)$

Proof.

$[BG, O] = \widehat{RO}(G)/\widehat{RU}(G)$ by mapping BG into the fibration sequence

$$\begin{array}{ccccccc} \Omega(U) & \longrightarrow & \Omega(U/O) & \longrightarrow & O & \longrightarrow & U. \\ \| & & \| & & & & \\ BU \times \mathbf{Z} & \longrightarrow & BO \times \mathbf{Z} & & & & \end{array}$$

since $[BG, U] = 0$. □

Proof.

$[BG, Sp] = \widehat{RSp}(G)/\widehat{RU}(G)$ by mapping BG into the fibration sequence

$$\begin{array}{ccccccc} \Omega(U) & \longrightarrow & \Omega(U/Sp) & \longrightarrow & Sp & \longrightarrow & U. \\ \| & & \| & & & & \\ BU \times \mathbf{Z} & \longrightarrow & BSp \times \mathbf{Z} & & & & \end{array}$$

since $[BG, U] = 0$.

□

Proof.

$[BG, U/Sp] = 0$ by mapping BG into the fibration sequence

$$U \longrightarrow U/Sp \longrightarrow BSp \times \mathbf{Z} \longrightarrow BU \times \mathbf{Z},$$

since $\widetilde{c} : \widehat{RSp}(G) \longrightarrow \widehat{RU}(G)$ is a monomorphism.

$[BG, U/O] = 0$ by mapping BG into the fibration sequence

$$U \longrightarrow U/O \longrightarrow BO \times \mathbf{Z} \longrightarrow BU \times \mathbf{Z},$$

since $c : \widehat{RO}(G) \longrightarrow \widehat{RU}(G)$ is a monomorphism.

□

Proof.

$[BG, O/U] = \widehat{RU}(G)/\widehat{RSp}(G)$ by mapping BG into the fibration sequence

$$\begin{array}{ccccccccccc} \Omega^2(O/U) & \longrightarrow & \Omega(U) & \longrightarrow & \Omega(O) & \longrightarrow & \Omega(O/U) & \longrightarrow & U & \longrightarrow & O. \\ \| & & \| & & \| & & \| & & & & \\ BSp \times \mathbf{Z} & \longrightarrow & BU \times \mathbf{Z} & \longrightarrow & O/U & \longrightarrow & U/Sp & & & & \end{array}$$

and using $[BG, U/Sp] = 0$.

□

Proof.

Finally, for Part (6) we see $[BG, Sp/U] = \widehat{RU}(G)/\widehat{RO}(G)$ by mapping BG into the fibration sequence

$$\begin{array}{ccccccccccc}
\Omega^2(Sp/U) & \longrightarrow & \Omega(U) & \longrightarrow & \Omega(Sp) & \longrightarrow & \Omega(Sp/U) & \longrightarrow & U & \longrightarrow & Sp. \\
\| & & \| & & \| & & \| & & & & \\
BO \times \mathbf{Z} & \longrightarrow & BU \times \mathbf{Z} & \longrightarrow & Sp/U & \longrightarrow & U/O & & & &
\end{array}$$

and using $[BG, U/O] = 0$.

□

Coefficients

Taking connective covers gives

$$ku^* = \mathbf{Z}[v]$$

and

$$ko^* = \frac{\mathbf{Z}[\eta, \alpha, \beta]}{(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)}.$$

These now relate cohomology and periodic K-theory:

$$\begin{array}{ccccc} & & ko & \xrightarrow{[\beta^{-1}]} & KO \\ & & \Big\downarrow c & & \Big\downarrow c \\ H\mathbf{Z} & \longleftarrow & ku & \xrightarrow[{[v^{-1}]}]{} & KU \end{array}$$

The cofiber sequence

$$\Sigma^2 ku \xrightarrow{v} ku \longrightarrow H\mathbf{Z} \longrightarrow \Sigma^3 ku$$

results in a Bockstein spectral sequence

$$H\mathbf{Z}^*(X)[v] \Longrightarrow ku^*(X)$$

with first differential

$$\overline{Q}_1 : H\mathbf{Z} \longrightarrow \Sigma^3 H\mathbf{Z}$$

Essentially the same as the Atiyah-Hirzebruch spectral sequence since

$$\cdots \xrightarrow{v} \Sigma^{2i} ku \xrightarrow{v} \cdots \xrightarrow{v} \Sigma^2 ku \xrightarrow{v} ku$$

is the Postnikov tower of ku.

Similarly, the cofiber sequence

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \xrightarrow{R} \Sigma^2 ko$$

results in a Bockstein spectral sequence

$$ku^*(X)[\eta] \Longrightarrow ko^*(X)$$

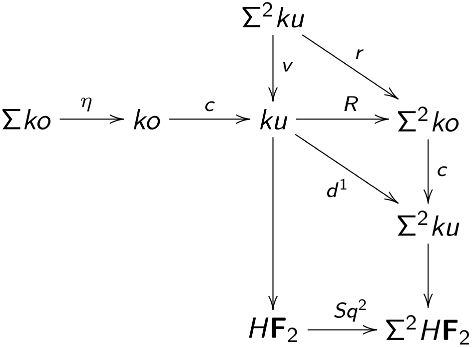
with first differential cR

- closely related to $1 + \tau$ in the periodic theory,
- and to Sq^2 in cohomology.

Better, since $\eta^3 = 0$,

- it collapses: $E^4 = E^\infty$
- at E^∞ it is concentrated on lines 0, 1 and 2.

The η-c-R sequence, its differential, and related operations



Since $H^*ku = \mathcal{A}//E(1)$ and $H^*ko = \mathcal{A}//\mathcal{A}(1)$, we have Adams spectral sequences

$$\mathrm{Ext}^{s,t}_{E(1)}(\mathbf{F}_2, H^*BG) \Longrightarrow ku^{-t+s}BG$$

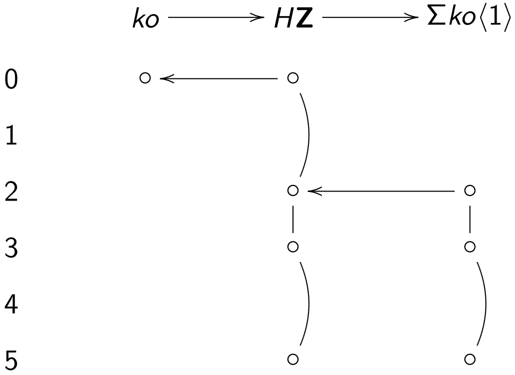
and

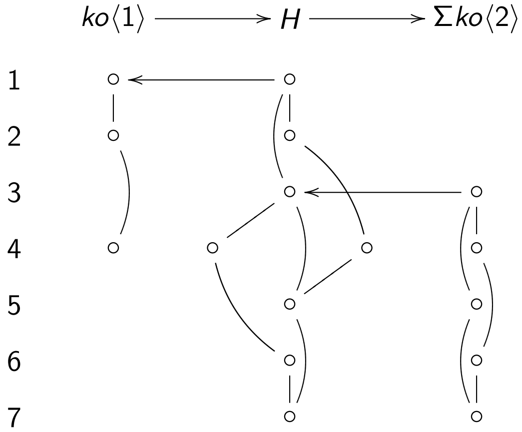
$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\mathbf{F}_2, H^*BG) \Longrightarrow ko^{-t+s}BG$$

(Associated graded cannot distinguish ku^*BG from ku^*_G.)
'Accounting device', destroys multiplicative information.

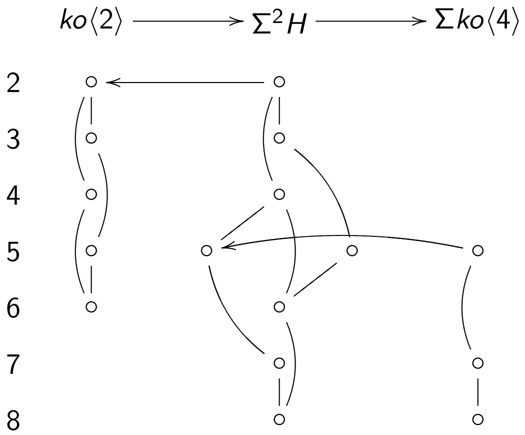
Typical use: show the Bott map acts monomorphically in a range beyond the edge of periodicity, so that relations can be accurately detected in periodic K-theory, which is determined by representation theory. Use the Adams spectral sequence to verify that the implications of these relations suffice.

Since H^*ko, $H^*H\mathbf{Z}$ and H^*H are all induced up from $\mathcal{A}(1)$, we can compute the cohomology of the Postnikov sections of ko in $\mathcal{A}(1)$-Mod and then tensor up to $\mathcal{A}$.



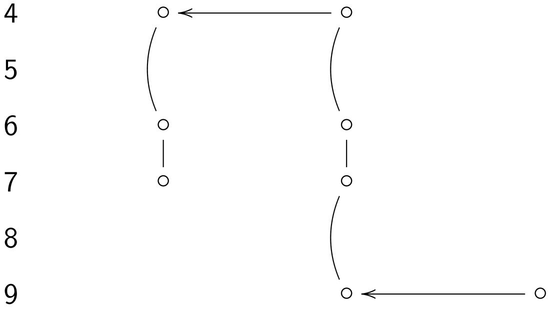


ko⟨1⟩
H
Σko⟨2⟩
1
2
3
4
5
6
7



ko⟨2⟩
Σ²H
Σko⟨4⟩
2
3
4
5
6
7
8

$$ko\langle 4\rangle \longrightarrow \Sigma^4 H\mathbf{Z} \longrightarrow \Sigma ko\langle 8\rangle$$



Corollary

The primary differentials in the Atiyah-Hirzebruch spectral sequences $H^p(X, ko_q) \Longrightarrow ko^{p-q}(X)$ *and* $H^p(X, KO_q) \Longrightarrow KO^{p-q}(X)$ *are:*

- $H^p(X, ko_{8i}) \xrightarrow{d_2} H^{p+2}(X, ko_{8i+1})$ *is* $H^p(X, \mathbf{Z}) \xrightarrow{Sq^2} H^{p+2}(X, \mathbf{Z}/2)$,
- $H^p(X, ko_{8i+1}) \xrightarrow{d_2} H^{p+2}(X, ko_{8i+2})$ *is* $H^p(X, \mathbf{Z}/2) \xrightarrow{Sq^2} H^{p+2}(X, \mathbf{Z}/2)$,
- $H^p(X, ko_{8i+2}) \xrightarrow{d_3} H^{p+3}(X, ko_{8i+4})$ *is* $H^p(X, \mathbf{Z}/2) \xrightarrow{Sq^3} H^{p+3}(X, \mathbf{Z})$, *and*
- $H^p(X, ko_{8i+4}) \xrightarrow{d_5} H^{p+5}(X, ko_{8i+8})$ *is* $H^p(X, \mathbf{Z}) \xrightarrow{Sq^5} H^{p+5}(X, \mathbf{Z})$.

The crude truncation $ku = KU[0,\infty)$ which we used in the non-equivariant case will not produce an interesting result in the equivariant case.
In particular it will not have Euler classes, and would not be complex orientable.

Solution: observe that any equivariant ku_G should sit in a commutative square

$$\begin{array}{ccccc} & & ku_G & \longrightarrow & KU_G \\ & & \downarrow & & \downarrow \\ F(EG_+, ku_G) & \xrightarrow{\simeq} & F(EG_+, \inf_1^G ku) & \longrightarrow & F(EG_+, \inf_1^G KU) \end{array}$$

and *define* ku_G to be the pullback.

Greenlees [JPAA 2004] showed this has good properties:

1. ku_G is a (strict) commutative ring G-spectrum.
2. If $H \subset G$ then $\mathrm{res}^G_H\, ku_G = ku_H$.
3. ku_G is a split ring G-spectrum.
4. $ku_G[v^{-1}] = KU_G$.
5. ku_G^* is Noetherian.
6. ku_G is complex orientable.
7. $ku_G^* \longrightarrow ku^* BG$ is completion
8. There is a local cohomology spectral sequence.

The same construction works in the real case.
We *define* ko_G to be the pullback.

$$\begin{array}{ccc} ko_G & \longrightarrow & KO_G \\ \downarrow & & \downarrow \\ F(EG_+, \inf_1^G ko) & \longrightarrow & F(EG_+, \inf_1^G KO) \end{array}$$

Calculational consequence

The coefficient rings sit in pullback squares

$$
\begin{array}{ccc}
ku_G^* & \longrightarrow & KU_G^* \\
\downarrow & & \downarrow \\
ku^*(BG) & \longrightarrow & KU^*(BG)
\end{array}
\qquad\qquad
\begin{array}{ccc}
ko_G^* & \longrightarrow & KO_G^* \\
\downarrow & & \downarrow \\
ko^*(BG) & \longrightarrow & KO^*(BG)
\end{array}
$$

More interesting, if $\widehat{G} = G \times C_2$, there is a $\widehat{G}$ spectrum $K\mathbf{R}$ representing G-equivariant periodic Real K-theory, in the sense of Atiyah.
There exists a C_2-map $K\mathbf{R} \longrightarrow K\mathbf{R}^G$ which is a C_2-equivalence, hence a $\widehat{G}$-map $\inf_{C_2}^{\widehat{G}} K\mathbf{R} \longrightarrow K\mathbf{R}$ of $\widehat{G}$-spectra which is a C_2-equivalence, so that

$$F(EG_+, K\mathbf{R}) \simeq F(EG_+, \inf_{C_2}^{\widehat{G}} K\mathbf{R})$$

as $\widehat{G}$-spectra.

We then *define* $k\mathbf{R}_G$ to be the pullback in $\widehat{G}$-spectra

$$\begin{array}{ccc} k\mathbf{R}_G & \longrightarrow & K\mathbf{R} \\ \downarrow & & \downarrow \\ F(EG_+, \inf_{C_2}^{\widehat{G}} k\mathbf{R}) & \longrightarrow & F(EG_+, \inf_{C_2}^{\widehat{G}} K\mathbf{R}) \end{array}$$

where the lower left $k\mathbf{R}$ is the connective cover of the C_2-spectrum $K\mathbf{R}$.

Theorem

As G-spectra, $k\mathbf{R}_G \simeq ku_G$ and $(k\mathbf{R}_G)^{C_2} \simeq ko_G$.

There are compatible Chern classes in cohomology, in connective and periodic K-theories, and in representation theory

$$c_i^H \longleftarrow\!\shortmid\; c_i^{ku} \;\longmapsto^{v^i}\; c_i^K \longleftarrow\!\shortmid\; c_i^R$$

$$\begin{array}{ccccccc}
H^{2i}(BU(n);\mathbf{Z}) & \longleftarrow & ku^{2i}BU(n) & \xrightarrow{v^i} & K^0BU(n) & \longleftarrow & R(U(n)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{2i}(BT^n;\mathbf{Z}) & \longleftarrow & ku^{2i}BT^n & \xrightarrow{v^i} & K^0BT^n & \longleftarrow & R(T^n)
\end{array}$$

which restrict to the symmetric polynomials in the Euler classes of the natural line bundles on a maximal torus $T^n \subset U(n)$.

Chern classes in representation theory

Definition

Let $n = \dim(V)$. Then

$$c_k^R(V) = \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} \Lambda^i(V)$$

Definition

The Chern (or 'gamma') filtration of representation theory:
$JU_i(G) \subset R(G)$ is the ideal generated by all products $c_{i_1}(V_1) \cdots c_{i_k}(V_k)$ with $i_1 + \cdots i_k \geq i$.

This is *multiplicative*:

$$JU_j(G) JU_k(G) \subset JU_{j+k}(G)$$

The formula for the c_k in terms of the Λ^i is the same as the formula for the Λ^k in terms of the c_i

$$\Lambda^k(V) = \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} c_i^R(V)$$

since this formula is the one that relates symmetric polynomials in variables $t_1, \ldots, t_n$ to symmmetric polynomials in $1 - t_1, \ldots, 1 - t_n$, and $t \mapsto 1 - t$ is an involution.

Modified Rees ring

Given a ring R and a multiplicative filtration $\mathcal{F} = \{R = F_0 \supset F_1 \supset \cdots\}$ the *Modified Rees ring*

$$MRees(R, \mathcal{F}) = \{ \sum_{i=-N}^{\infty} r_i t^i \mid r_{-i} \in F_i \text{ for } i > 0\}$$

The usual Rees ring construction uses $F_i = I^i$ for an ideal $I \subset R$.

If $\mathcal{C}$ is the Chern filtration of $R(G)$ then

$$MRees(RU(G)) := MRees(RU(G), \mathcal{C})$$

is a very good aproximation to ku_G^*.

The Complex Case

Lemma

$JU_1(G)$ is the augmentation ideal $JU(G)$, consisting of representations of virtual dimension 0.

Proof.

Since JU is generated by first Chern classes, $c_1^R(V) = \dim(V) - V$,

$$JU \subset JU_1.$$

□

Proof.

(Cont.) Conversely, for $k > 0$,

$$\begin{aligned}
\dim(c_k^R(V)) &= \dim \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} \Lambda^i(V) \\
&= \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} \dim \Lambda^i(V) \\
&= \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} \binom{n}{i} \\
&= \binom{n}{n-k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \\
&= 0
\end{aligned}$$

so $JU_1 \subset JU$. □

Lemma

JU_2 consists of representations of virtual dimension 0 and virtual determinant 1.

For the defining representation of $U(n)$,

$$\begin{aligned} \det = \Lambda^n &= 1 - c_1^R + \cdots + (-1)^n c_n^R \\ &= 1 - v c_1^{ku} + \cdots + (-v)^n c_n^{ku}. \end{aligned}$$

Since $SU(n)$ is the fiber of $\det : U(n) \longrightarrow U(1)$,

$$\begin{aligned} ku^* BSU(n) &= ku^* BU(n)/(c_1^{ku}(\det) \\ &= ku^*[[c_1, \ldots, c_n]]/((1 - \Lambda^n)/v) \\ &= ku^*[[c_1, \ldots, c_n]]/(c_1 - vc_2 + v^2 c_3 - \cdots + (-v)^{n-1} c_n). \end{aligned}$$

('Unnaturally' isomorphic to $ku^*[[c_2, \ldots, c_n]]$.)

JU_2 is generated by products of Chern classes and by $c_i(V)$ for $i \geq 2$.

Let JU_2' be the ideal of virtual dimension 0, determinant 1 representations, i.e., those whose classifing map lifts over $BSU \longrightarrow BU \longrightarrow BU \times \mathbf{Z}$.

For $V - W \in JU_2'$, let

$$\delta = \det(V) = \det(W) \qquad \text{and} \qquad n = \dim(V) = \dim(W).$$

$\delta \in R(G)^\times$, so it suffices to show $V\delta^{-1} - W\delta^{-1} \in JU_2$.
So we may assume $\delta = 1$.

Since $V - W = (n - W) - (n - V)$, it suffices to show

$$c_1(W) = n - W \in JU_2.$$

$\det(W) = 1$ so the representation W lifts to $SU(n)$.

In $ku^*BSU(n)$,

$$c_1 = vc_2 - v^2c_3 + \cdots \pm v^{n-1}c_n$$

and it follows that $JU_2' \subset JU_2$.

Conversely, we must show that any Chern class $c_k(V)$, $k > 1$, and any product $c_1(V)c_1(W)$, have dimension 0 and determinant 1. The first was shown already. For the second, recall that

$$\det(kV) = (\det(V))^k \qquad \text{and} \qquad \det(\Lambda^i(V)) = (\det(V))^{\binom{n-1}{i-1}}$$

for $i > 0$. Thus

$$\begin{aligned} \det(c_k(V)) &= \prod_{i=1}^{k} (\det \Lambda^i(V))^{\left((-1)^i \binom{n-i}{n-k}\right)} \\ &= \prod_{i=1}^{k} (\det V)^{\left((-1)^i \binom{n-i}{n-k}\binom{n-1}{i-1}\right)} \end{aligned}$$

This is $\det(V)$ raised to the power

$$\sum_{i=1}^{k} (-1)^i \binom{n-i}{n-k}\binom{n-1}{i-1} = \binom{n-1}{k-1} \sum_{i=1}^{k} (-1)^i \binom{k-1}{i-1} = 0,$$

since $k - 1 > 0$.

Finally, if $m = \dim(V)$ and $n = \dim(W)$. Then

$$\dim(c_1(V)c_1(W)) = \dim((m - V)(n - W)) = 0$$

and

$$\det(m - V)(n - W)) = \det(mn - nV - mW + VW) = 1$$

since

$$\det(VW) = (\det(V))^n(\det(W))^m.$$

□

Theorem

$ku^*_G \longrightarrow KU^*_G$ is a monomorphism in codegrees ≤ 5 and

$$ku^i_G = \begin{cases} 0 & i \leq 0 \quad \text{odd} \\ RU(G) & i \leq 0 \quad \text{even} \\ 0 & i = 1 \\ JU(G) & i = 2 \\ 0 & i = 3 \\ JU_2(G) & i = 4 \\ 0 & i = 5 \end{cases}$$

Proof.

Compare the Atiyah-Hirzebruch spectral sequences

$$\begin{array}{ccc} H^p(BG, ku_q) & \Longrightarrow & ku^{p-q}(BG) \\ \downarrow & & \downarrow \\ H^p(BG, KU_q) & \Longrightarrow & KU^{p-q}(BG) \end{array}$$

Proof.

(Cont.) The only differential which could affect the difference between them is

$$d_3 : H^2(BG, KU_{-2}) \longrightarrow H^5(BG, KU_0).$$

But every element of $H^2(BG, KU_{-2})$ is a first Chern class, and these survive by the universal example.

This gives $ku^5 BG = 0$ and shows that $ku^4 BG$ is the kernel of the map $KU_G^4 \longrightarrow H^2(BG; \mathbf{Z}) \otimes H^0(BG, \mathbf{Z})$ induced by the Postnikov section

$$BU \times \mathbf{Z} \xrightarrow{B\det \times 1} BU(1) \times \mathbf{Z} \simeq K(\mathbf{Z}, 2) \times K(\mathbf{Z}, 0).$$

By the Lemma, this is exactly $\widehat{JU}_2(G)$. The pullback diagram then gives the uncompleted results.

□

Beyond this, complications set in.

Theorem

There are exact sequences

$$0 \longrightarrow H^3BG \xrightarrow{\overline{Q}_1} ku^6BG \xrightarrow{v} ku^4BG = \widehat{JU}_2(G) \longrightarrow H^4BG \xrightarrow{\overline{Q}_1} ku^7BG \longrightarrow 0$$

and

$$0 \longrightarrow H^5BG \xrightarrow{\overline{Q}_1} ku^8BG \xrightarrow{v} ku^6BG \longrightarrow H^6BG \xrightarrow{\overline{Q}_1} ku^9BG \longrightarrow \cdots$$ □

Proof.

Use $\Sigma^2 ku \xrightarrow{v} ku \longrightarrow H\mathbf{Z}$. □

The Real Case

Theorem

$ko_G^* \longrightarrow KO_G^*$ is a monomorphism in codegrees ≤ 7, and

$$ko_G^i = \begin{cases} 0 & i = 1 \\ JU(G)/JO(G) \subset RU(G)/RO(G) & i = 2 \\ JSp(G)/JU(G) \subset RSp(G)/RU(G) & i = 3 \\ JSp(G) \subset RSp(G) & i = 4 \\ 0 & i = 5 \\ JU_2(G)/JSp(G) \subset RU(G)/RSp(G) & i = 6 \\ JSpin(G)/JU_2(G) \subset RO(G)/RU(G) & i = 7 \end{cases}$$

As in the complex case $ko_G^i \longrightarrow KO_G^i$ is often a monomorphism for $i = 8$ or 9. There is an exact sequence

$$JSO \xrightarrow{\beta w_2} H^3(BG; \mathbf{Z}) \longrightarrow ko_G^8 \longrightarrow KO_{-8}^G = RO(G)$$

and $ko_G^9 = 0$ iff the two maps

$$JSO \xrightarrow{w_2} H^2(BG; \mathbf{Z}/2) \text{ and } RSpin(G) \xrightarrow{p_1/2} H^4(BG; \mathbf{Z})$$

are monomorphisms.

The argument goes as in the real case, but the difference between the Atiyah-Hirzebruch spectral sequences for ko^*BG and KO^*BG is more complicated. The analysis is helped by the following.

Theorem

The first nine spaces in the spectrum ko and the Moore-Postnikov factorization of $ko \longrightarrow KO$ are

0 $BO \times \mathbf{Z} \longrightarrow BO \times \mathbf{Z}$

1 $U/O \longrightarrow U/O$

2 $Sp/U \longrightarrow Sp/U$

3 $Sp \longrightarrow Sp$

4 $BSp \longrightarrow BSp \times \mathbf{Z}$

5 $SU/Sp \longrightarrow U/Sp$

6 $Spin/SU \longrightarrow SO/U \longrightarrow O/U$

7 $String \longrightarrow Spin \longrightarrow SO \longrightarrow O$

8 $BString \longrightarrow BSpin \longrightarrow BSO \longrightarrow BO \longrightarrow BO \times \mathbf{Z}$

This is proved at the same time as we identify maps from BG into various of these homogeneous spaces. Denote the connective covers of O by $SO = O\langle 1\rangle$, $Spin = O\langle 3\rangle$, and $String = O\langle 7\rangle$. Then

Theorem

For the 0, 2, and 6-connected covers of O and associated homogeneous spaces, we have:

1. $\Omega(SO/U) = U/Sp$ and $[BG, SO/U] = \widehat{JU}(G)/\widehat{JSp}(G)$
2. $\Omega(Spin/SU) = SU/Sp$ and $[BG, Spin/SU] = \widehat{JU_2}(G)/\widehat{JSp}(G)$
3. $\Omega SO = O/U$ and $[BG, SO] = \widehat{JO}(G)/\widehat{JU}(G)$
4. $\Omega Spin = SO/U$ and $[BG, Spin] = \widehat{JSO}(G)/\widehat{JU}(G)$
5. $\Omega String = Spin/SU$ and $[BG, String] = \widehat{JSpin}(G)/\widehat{JU_2}(G)$

The most interesting of these is *String*. The relevant diagram in this case is

$$\begin{array}{ccccc}
\Sigma^2 H\mathbf{Z} & \longrightarrow & \Sigma^2 H\mathbf{Z}/2 & \xrightarrow{\beta} & \Sigma^3 H\mathbf{Z} \\
\uparrow & & \uparrow{\scriptstyle w_2} & & \uparrow \\
BU & \longrightarrow & BSO & \longrightarrow & Spin \\
\uparrow & & \uparrow & & \uparrow \\
BSU & \longrightarrow & BSpin & \longrightarrow & String
\end{array}$$

The latter fibre sequence shows that $\Omega(String) = Spin/SU$ and that $[BG, String]$ is $\widehat{JSpin}(G)/\widehat{JU}_2(G)$, since $[BG, B^2SU] = [BG, U\langle 5\rangle] = 0$.

To get the fiber sequence in the middle row of the previous diagram, consider:

$$\begin{array}{ccccc} & & \Sigma H\mathbf{Z}/2 & = & \Sigma H\mathbf{Z}/2 \\ & & \uparrow{\scriptstyle w_1} & & \uparrow{\scriptstyle \Omega w_2} \\ BU & \longrightarrow & BO & \longrightarrow & SO \\ \| & & \uparrow & & \uparrow \\ BU & \longrightarrow & BSO & \longrightarrow & Spin \end{array}$$

The latter fibre sequence shows that $[BG, Spin] = \widehat{JSO}(G)/\widehat{JU}(G)$ since, again, $[BG, B^2U] = [BG, SU] = 0$.

End of Part One