Characteristic Classes in K-Theory
General Theory

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Topology Seminar
Universitetet i Bergen
10 August 2011

Outline

Chern classes in general

If

$$\alpha : G \longrightarrow U(n)$$

we have compatible Chern classes

$$\begin{array}{ccccccc} c_i^H(\alpha) & \longleftarrow & c_i^{ku}(\alpha) & \xrightarrow{v^i} & c_i^K(\alpha) & \longleftarrow & c_i^R(\alpha) \\ H^{2i}BG & \longleftarrow & ku^{2i}BG & \xrightarrow{v^i} & K^0BG & \longleftarrow & R(G) \end{array}$$

defined by universal example.

All but the c_i^R are familiar, and those we'll derive from the universal example.

Under restriction to the maximal torus $T^n \subset U(n)$, we have

$$\Lambda_i \mapsto \sigma_i(l_1, \ldots, l_n).$$

We want

$$c_i = c_i(\Lambda_1) \mapsto \sigma_i(c_1(l_1), \ldots, c_1(l_n))$$

Then we'll have

$$c_i^H \longleftarrow\!\mid\; c_i^{ku} \;\overset{v^i}{\longmapsto}\; c_i^K \longleftarrow\!\mid\; c_i^R$$

$$\begin{array}{ccccccc}
H^{2i}(BU(n);\mathbf{Z}) & \longleftarrow & ku^{2i}BU(n) & \xrightarrow{v^i} & K^0BU(n) & \longleftarrow & R(U(n)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{2i}(BT^n;\mathbf{Z}) & \longleftarrow & ku^{2i}BT^n & \xrightarrow{v^i} & K^0BT^n & \longleftarrow & R(T^n)
\end{array}$$

Chern classes in representation theory

Definition

Let $n = \dim(V)$. Then

$$c_k^R(V) = \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} \Lambda^i(V)$$

That this restricts as desired follows from the requirement that

$$\sum_0^n c_i t^{n-i} = \prod_1^n (t + c_1(l_i)) = \prod_1^n (t + 1 - l_i) = \sum_0^n (-1)^i (t+1)^{n-i} \Lambda_i$$

Observe that, since $t \mapsto 1 - t$ is an involution, we also have

$$\Lambda^k(V) = \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} c_i^R(V)$$

Definition

The Chern (or 'gamma') filtration of representation theory:
$JU_i(G) \subset R(G)$ is the ideal generated by all products $c_{i_1}(V_1)\cdots c_{i_k}(V_k)$ with $i_1 + \cdots i_k \geq i$.

This is *multiplicative*:

$$JU_j(G)JU_k(G) \subset JU_{j+k}(G)$$

Modified Rees ring

Given a ring R and a multiplicative filtration $\mathcal{F} = \{R = F_0 \supset F_1 \supset \cdots\}$ the *Modified Rees ring*

$$MRees(R, \mathcal{F}) = \{ \sum_{i=-N}^{\infty} r_i t^i \mid r_{-i} \in F_i \text{ for } i > 0\}$$

The usual Rees ring construction uses $F_i = I^i$ for an ideal $I \subset R$.

If $\mathcal{C}$ is the Chern filtration of $R(G)$ then

$$MRees(RU(G)) := MRees(RU(G), \mathcal{C})$$

is a very good approximation to the image of ku_G^* in KU_G^*.

The construction

The crude truncation $ku = KU[0, \infty)$ used in the non-equivariant case will not produce an interesting result in the equivariant case.
In particular it will not have Euler classes, and would not be complex orientable.

Solution: observe that any equivariant ku_G should sit in a commutative square

$$\begin{array}{ccccc} & & ku_G & \longrightarrow & KU_G \\ & & \downarrow & & \downarrow \\ F(EG_+, ku_G) & \xrightarrow{\simeq} & F(EG_+, \inf_1^G ku) & \longrightarrow & F(EG_+, \inf_1^G KU) \end{array}$$

and *define* ku_G to be the pullback.

Greenlees [JPAA 2004] showed this has good properties:

1. ku_G is a (strict) commutative ring G-spectrum.
2. If $H \subset G$ then $\mathrm{res}^G_H ku_G = ku_H$.
3. ku_G is a split ring G-spectrum.
4. $ku_G[v^{-1}] = KU_G$.
5. ku_G^* is Noetherian.
6. ku_G is complex orientable.
7. $ku_G^* \longrightarrow ku^*BG$ is completion
8. There is a local cohomology spectral sequence.

The same construction works in the real case.
We *define* ko_G to be the pullback.

$$\begin{array}{ccc} ko_G & \longrightarrow & KO_G \\ \downarrow & & \downarrow \\ F(EG_+, \inf_1^G ko) & \longrightarrow & F(EG_+, \inf_1^G KO) \end{array}$$

Calculational consequence

The coefficient rings sit in pullback squares

$$\begin{array}{ccc} ku_G^* & \longrightarrow & KU_G^* \\ \downarrow & & \downarrow \\ ku^*(BG) & \longrightarrow & KU^*(BG) \end{array} \qquad \begin{array}{ccc} ko_G^* & \longrightarrow & KO_G^* \\ \downarrow & & \downarrow \\ ko^*(BG) & \longrightarrow & KO^*(BG) \end{array}$$

The coefficients

We easily compute the ku_G^i for i near 0 in terms of dimensions and determinants, but our goal is to relate them to the Chern filtration, so we start by studying that.

Lemma

$JU_1(G)$ is the augmentation ideal $JU(G)$, consisting of representations of virtual dimension 0.

Proof.

Since JU is generated by first Chern classes, $c_1^R(V) = \dim(V) - V$,

$$JU \subset JU_1.$$

□

Proof.

(Cont.) Conversely, for $k > 0$,

$$\begin{aligned} \dim(c_k^R(V)) &= \dim \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} \Lambda^i(V) \\ &= \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} \dim \Lambda^i(V) \\ &= \sum_{i=0}^{k} (-1)^i \binom{n-i}{n-k} \binom{n}{i} \\ &= \binom{n}{n-k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \\ &= 0 \end{aligned}$$

so $JU_1 \subset JU$. □

Lemma

JU_2 consists of representations of virtual dimension 0 and virtual determinant 1.

For the defining representation of $U(n)$,

$$\begin{aligned} \det = \Lambda^n &= 1 - c_1^R + \cdots + (-1)^n c_n^R \\ &= 1 - v c_1^{ku} + \cdots + (-v)^n c_n^{ku}. \end{aligned}$$

Since $SU(n)$ is the fiber of $\det : U(n) \longrightarrow U(1)$,

$$\begin{aligned} ku^* BSU(n) &= ku^* BU(n)/(c_1^{ku}(\det) \\ &= ku^*[[c_1, \ldots, c_n]]/((1 - \Lambda^n)/v) \\ &= ku^*[[c_1, \ldots, c_n]]/(c_1 - v c_2 + v^2 c_3 - \cdots + (-v)^{n-1} c_n). \end{aligned}$$

('Unnaturally' isomorphic to $ku^*[[c_2, \ldots, c_n]]$.)

JU_2 is generated by products of Chern classes and by $c_i(V)$ for $i \geq 2$.

Let JU_2' be the ideal of virtual dimension 0, determinant 1 representations, i.e., those whose classifing map lifts over $BSU \longrightarrow BU \longrightarrow BU \times \mathbf{Z}$.

For $V - W \in JU_2'$, let

$$\delta = \det(V) = \det(W) \qquad \text{and} \qquad n = \dim(V) = \dim(W).$$

$\delta \in R(G)^\times$, so it suffices to show $V\delta^{-1} - W\delta^{-1} \in JU_2$.

So we may assume $\delta = 1$.

Since $V - W = (n - W) - (n - V)$, it suffices to show

$$c_1(W) = n - W \in JU_2.$$

$\det(W) = 1$ so the representation W lifts to $SU(n)$.

In $ku^*BSU(n)$,

$$c_1 = vc_2 - v^2c_3 + \cdots \pm v^{n-1}c_n$$

and it follows that $JU_2' \subset JU_2$.

Conversely, we must show that any Chern class $c_k(V)$, $k > 1$, and any product $c_1(V)c_1(W)$, have dimension 0 and determinant 1. The first was shown already. For the second, recall that

$$\det(kV) = (\det(V))^k \qquad \text{and} \qquad \det(\Lambda^i(V)) = (\det(V))^{\binom{n-1}{i-1}}$$

for $i > 0$. Thus

$$\begin{aligned} \det(c_k(V)) &= \prod_{i=1}^{k} (\det \Lambda^i(V))^{\left((-1)^i \binom{n-i}{n-k}\right)} \\ &= \prod_{i=1}^{k} (\det V)^{\left((-1)^i \binom{n-i}{n-k}\binom{n-1}{i-1}\right)} \end{aligned}$$

This is $\det(V)$ raised to the power

$$\sum_{i=1}^{k} (-1)^i \binom{n-i}{n-k}\binom{n-1}{i-1} = \binom{n-1}{k-1} \sum_{i=1}^{k} (-1)^i \binom{k-1}{i-1} = 0,$$

since $k - 1 > 0$.

Finally, if $m = \dim(V)$ and $n = \dim(W)$. Then

$$\dim(c_1(V)c_1(W)) = \dim((m - V)(n - W)) = 0$$

and

$$\det(m - V)(n - W)) = \det(mn - nV - mW + VW) = 1$$

since

$$\det(VW) = (\det(V))^n(\det(W))^m.$$

□

Theorem

$ku_G^* \longrightarrow KU_G^*$ is a monomorphism in codegrees ≤ 5 and

$$ku_G^i = \begin{cases} 0 & i \leq 0 \quad \text{odd} \\ RU(G) & i \leq 0 \quad \text{even} \\ 0 & i = 1 \\ JU(G) & i = 2 \\ 0 & i = 3 \\ JU_2(G) & i = 4 \\ 0 & i = 5 \end{cases}$$

Use the long exact sequence induced by the cofiber sequence

$$\Sigma^2 ku \longrightarrow ku \longrightarrow H\mathbf{Z} \longrightarrow \Sigma^3 ku$$

Proof.

We immediately see that $ku_G^1 \xrightarrow{v} ku_G^{-1} = 0$ is an isomorphism.

Restriction to the trivial group shows that $ku_G^0 \longrightarrow H^0BG$ computes the dimension of the virtual representation, so that $ku_G^2 = JU(G) = JU_1(G)$.

We then have

$$H^0BG \xrightarrow{0} ku_G^3 \xrightarrow{v} ku_G^1 \longrightarrow H^1(BG, \mathbf{Z}) = 0$$

so that $ku_G^3 = ku_G^1 = 0$.

$H^1BG = 0$ also implies that $(ku_G^4 \xrightarrow{v} ku_G^2) = \ker(ku_G^2 \longrightarrow H^2BG)$ so that $ku_G^4 = JU_2(G)$.

Since every element of $H^2(BG, \mathbf{Z})$ is a first Chern class, $H^2BG \longrightarrow ku_G^5$ is zero since it is for $G = U(1)$, and hence $ku_G^5 = ku^3G = 0$. □

Beyond this, complications set in.

Theorem

There are exact sequences

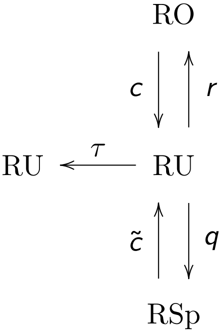
$$0 \longrightarrow H^3BG \xrightarrow{\overline{Q}_1} ku_G^6 \xrightarrow{v} ku_G^4 = JU_2(G)$$
$$\longrightarrow H^4BG \xrightarrow{\overline{Q}_1} ku_G^7 \longrightarrow 0$$

and

$$0 \longrightarrow H^5BG \xrightarrow{\overline{Q}_1} ku_G^8 \xrightarrow{v} ku_G^6 \longrightarrow H^6BG \xrightarrow{\overline{Q}_1} ku_G^9 \longrightarrow \cdots \qquad \square$$

Representation Rings

Restriction, induction and conjugation induce natural transformations between the real, complex, and quaternionic representation rings:



$$rc = 2$$

$$cr = 1 + \tau \qquad r\tau = r$$

$$\tau c = c \qquad \tau\tilde{c} = \tilde{c}$$

$$\tilde{c}q = 1 + \tau \qquad q\tau = q$$

$$q\tilde{c} = 2$$

Representation Rings

We may describe all these maps quite explicitly.
For any compact Lie group, we may choose

- irreducible real representations U_i,
- irreducible complex representations V_j, and
- irreducible quaternionic representations W_k

so that

- $RU = \mathbf{Z}\langle cU_i \rangle \oplus \mathbf{Z}\langle V_j, \tau V_j \rangle \oplus \mathbf{Z}\langle \tilde{c} W_k \rangle$
- $RO = \mathbf{Z}\langle U_i \rangle \oplus \mathbf{Z}\langle rV_j \rangle \oplus \mathbf{Z}\langle rqW_k \rangle$
- $RSp = \mathbf{Z}\langle qcU_i \rangle \oplus \mathbf{Z}\langle qV_j \rangle \oplus \mathbf{Z}\langle W_k \rangle$

Atiyah-Segal Theorem

The Atiyah-Segal Theorem asserts that the natural map

$$S = Ee_+ \longleftarrow EG_+$$

induces completion at the augmentation ideal

- $KU_G^0 = RU(G) \longrightarrow KU_G^0(EG_+) = KU^0(BG)$
- $KO_G^0 = RO(G) \longrightarrow KO_G^0(EG_+) = KO^0(BG)$
- $KSp_G^0 = RSp(G) \longrightarrow KSp_G^0(EG_+) = KSp^0(BG)$

They also show that $[BG, U] = 0 = [BG, O] = [BG, Sp]$.

Write $\widehat{RO}$ for $RO_I^\wedge$ and similarly for related rings and ideals.

Using representations of G on Clifford modules, Atiyah, Bott and Shapiro show:

Theorem

1. $KU_G^* = RU(G)[v, v^{-1}]$.
2. $KO_G^* = RO^*(G)[\beta, \beta^{-1}]$ *where*

$$
\begin{array}{rcll}
RO^0 & = & RO & \cong \mathbf{Z}\{U_i, rV_j, r\tilde{c}W_k\} \\
RO^{-1} & = & RO/RU & \cong \mathbf{F}_2\{U_i\} \\
RO^{-2} & = & RU/RSp & \cong \mathbf{F}_2\{cU_i\} \oplus \mathbf{Z}\{\overline{V}_j\} \\
RO^{-3} & = & 0 & \\
RO^{-4} & = & RSp & \cong \mathbf{Z}\{qcU_i, qV_j, W_k\} \\
RO^{-5} & = & RSp/RU & \cong \mathbf{F}_2\{W_k\} \\
RO^{-6} & = & RU/RO & \cong \mathbf{Z}\{\overline{V}_j\} \oplus \mathbf{F}_2\{\tilde{c}W_k\} \\
RO^{-7} & = & 0 &
\end{array}
$$

Coefficients

The action of the coefficients,

$$KU^* = \mathbf{Z}[v, v^{-1}]$$

and

$$KO^* = \frac{\mathbf{Z}[\eta, \alpha, \beta, \beta^{-1}]}{(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)}$$

with $v \in KU^{-2}$, $\eta \in KO^{-1}$, $\alpha \in KO^{-4}$, and $\beta \in KO^{-8}$,

coincide with natural maps in representation theory.

For example, η induces the natural quotients

$$RO \longrightarrow RO/RU \qquad \text{and} \qquad RSp \longrightarrow RSp/RU$$

and the evident inclusions

$$RO/RU \longrightarrow RU/RSp \qquad \text{and} \qquad RSp/RU \longrightarrow RU/RO.$$

On the level of Clifford algebras, multiplication by η is complexification.

Similarly multiplication by α is quaternionification. Precisely, it is

- $qc : RO \longrightarrow RSp$ in degrees 0 mod 8
- $r\tilde{c} : RSp \longrightarrow RO$ in degrees 4 mod 8
- multiplication by 2,

  $$\mathbf{F}_2\{cU_i\} \oplus \mathbf{Z}\{\overline{V}_j\} \longrightarrow \mathbf{Z}\{\overline{V}_j\} \oplus \mathbf{F}_2\{\tilde{c}W_k\}$$

  and

  $$\mathbf{Z}\{\overline{V}_j\} \oplus \mathbf{F}_2\{\tilde{c}W_k\} \longrightarrow \mathbf{F}_2\{cU_i\} \oplus \mathbf{Z}\{\overline{V}_j\}$$

  in degrees 2 mod 4,
- and 0 in odd degrees.

Bott Periodicity

There is a more elementary deduction of the values of the periodic completed theory, which we will use to analyze the connective case.

It uses just the spaces in Bott periodicity, together with the isomorphisms

$$[BG, BO \times \mathbf{Z}] = \widehat{RO}(G), \qquad [BG, BSp \times \mathbf{Z}] = \widehat{RSp}(G),$$

and

$$[BG, BU \times \mathbf{Z}] = \widehat{RU}(G), \qquad [BG, U] = 0.$$

Recall that Bott periodicity says that starting with $BO \times \mathbf{Z}$ and repeatedly taking loops gives

- O
- O/U
- U/Sp
- $BSp \times \mathbf{Z}$
- Sp
- Sp/U
- U/O, and then
- $BO \times \mathbf{Z}$ again.

Theorem

The values of $[BG, -]$ on the infinite loop spaces above are as follows:

1. $[BG, O] = \widehat{RO}(G)/\widehat{RU}(G)$
2. $[BG, Sp] = \widehat{RSp}(G)/\widehat{RU}(G)$
3. $[BG, U/Sp] = 0$
4. $[BG, U/O] = 0$
5. $[BG, O/U] = \widehat{RU}(G)/\widehat{RSp}(G)$
6. $[BG, Sp/U] = \widehat{RU}(G)/\widehat{RO}(G)$

Proof.

$[BG, O] = \widehat{RO}(G)/\widehat{RU}(G)$ by mapping BG into the fibration sequence

$$\begin{array}{ccccccc} \Omega(U) & \longrightarrow & \Omega(U/O) & \longrightarrow & O & \longrightarrow & U. \\ \| & & \| & & & & \\ BU \times \mathbf{Z} & \longrightarrow & BO \times \mathbf{Z} & & & & \end{array}$$

since $[BG, U] = 0$. □

Proof.

$[BG, Sp] = \widehat{RSp}(G)/\widehat{RU}(G)$ by mapping BG into the fibration sequence

$$\begin{array}{ccccccc}
\Omega(U) & \longrightarrow & \Omega(U/Sp) & \longrightarrow & Sp & \longrightarrow & U. \\
\| & & \| & & & & \\
BU \times \mathbf{Z} & \longrightarrow & BSp \times \mathbf{Z} & & & &
\end{array}$$

since $[BG, U] = 0$.

□

Proof.

$[BG, U/Sp] = 0$ by mapping BG into the fibration sequence

$$U \longrightarrow U/Sp \longrightarrow BSp \times \mathbf{Z} \longrightarrow BU \times \mathbf{Z},$$

since $\widetilde{c} : \widehat{RSp}(G) \longrightarrow \widehat{RU}(G)$ is a monomorphism.

$[BG, U/O] = 0$ by mapping BG into the fibration sequence

$$U \longrightarrow U/O \longrightarrow BO \times \mathbf{Z} \longrightarrow BU \times \mathbf{Z},$$

since $c : \widehat{RO}(G) \longrightarrow \widehat{RU}(G)$ is a monomorphism. □

Proof.

$[BG, O/U] = \widehat{RU}(G)/\widehat{RSp}(G)$ by mapping BG into the fibration sequence

$$\begin{array}{ccccccccccc}
\Omega^2(O/U) & \longrightarrow & \Omega(U) & \longrightarrow & \Omega(O) & \longrightarrow & \Omega(O/U) & \longrightarrow & U & \longrightarrow & O. \\
\| & & \| & & \| & & \| & & & & \\
BSp \times \mathbf{Z} & \longrightarrow & BU \times \mathbf{Z} & \longrightarrow & O/U & \longrightarrow & U/Sp & & & &
\end{array}$$

and using $[BG, U/Sp] = 0$.

□

Proof.

Finally, for Part (6) we see $[BG, Sp/U] = \widehat{RU}(G)/\widehat{RO}(G)$ by mapping BG into the fibration sequence

$$\begin{array}{ccccccccccc}
\Omega^2(Sp/U) & \longrightarrow & \Omega(U) & \longrightarrow & \Omega(Sp) & \longrightarrow & \Omega(Sp/U) & \longrightarrow & U & \longrightarrow & Sp. \\
\| & & \| & & \| & & \| & & & & \\
BO \times \mathbf{Z} & \longrightarrow & BU \times \mathbf{Z} & \longrightarrow & Sp/U & \longrightarrow & U/O & & & &
\end{array}$$

and using $[BG, U/O] = 0$.

□

The connective case

Theorem

$ko_G^* \longrightarrow KO_G^*$ is a monomorphism in codegrees ≤ 7, and

$$ko_G^i = \begin{cases} 0 & i = 1 \\ JU/JO \subset RU/RO & i = 2 \\ JSp/JU \subset RSp/RU & i = 3 \\ JSp \subset RSp & i = 4 \\ 0 & i = 5 \\ JU_2/JSp \subset RU/RSp & i = 6 \\ JSpin/JU_2 \subset RO/RU & i = 7 \end{cases}$$

The argument goes as in the complex case using the following.

Theorem

The first nine spaces in the spectrum ko and the Moore-Postnikov factorization of $ko \longrightarrow KO$ *are*

0 $BO \times \mathbf{Z} \longrightarrow BO \times \mathbf{Z}$

1 $U/O \longrightarrow U/O$

2 $Sp/U \longrightarrow Sp/U$

3 $Sp \longrightarrow Sp$

4 $BSp \longrightarrow BSp \times \mathbf{Z}$

5 $SU/Sp \longrightarrow U/Sp$

6 $Spin/SU \longrightarrow SO/U \longrightarrow O/U$

7 $String \longrightarrow Spin \longrightarrow SO \longrightarrow O$

8 $BString \longrightarrow BSpin \longrightarrow BSO \longrightarrow BO \longrightarrow BO \times \mathbf{Z}$

This is proved at the same time as we identify maps from BG into various of these homogeneous spaces. Denote the connective covers of O by $SO = O\langle 1\rangle$, $Spin = O\langle 3\rangle$, and $String = O\langle 7\rangle$. Then

Theorem

For the 0, 2, and 6-connected covers of O and associated homogeneous spaces, we have:

1. $\Omega(SO/U) = U/Sp$ and $[BG, SO/U] = \widehat{JU}(G)/\widehat{JSp}(G)$
2. $\Omega(Spin/SU) = SU/Sp$ and $[BG, Spin/SU] = \widehat{JU_2}(G)/\widehat{JSp}(G)$
3. $\Omega SO = O/U$ and $[BG, SO] = \widehat{JO}(G)/\widehat{JU}(G)$
4. $\Omega Spin = SO/U$ and $[BG, Spin] = \widehat{JSO}(G)/\widehat{JU}(G)$
5. $\Omega String = Spin/SU$ and $[BG, String] = \widehat{JSpin}(G)/\widehat{JU_2}(G)$

The most interesting of these is *String*. The relevant diagram in this case is

$$\begin{array}{ccccc}
\Sigma^2 H\mathbf{Z} & \longrightarrow & \Sigma^2 H\mathbf{Z}/2 & \xrightarrow{\beta} & \Sigma^3 H\mathbf{Z} \\
\uparrow & & \uparrow{\scriptstyle w_2} & & \uparrow \\
BU & \longrightarrow & BSO & \longrightarrow & Spin \\
\uparrow & & \uparrow & & \uparrow \\
BSU & \longrightarrow & BSpin & \longrightarrow & String
\end{array}$$

The latter fibre sequence shows that $\Omega(String) = Spin/SU$ and that $[BG, String]$ is $\widehat{JSpin}(G)/\widehat{JU}_2(G)$, since $[BG, B^2SU] = [BG, U\langle 5\rangle] = 0$.

To get the fiber sequence in the middle row of the previous diagram, consider:

$$\begin{array}{ccccc}
 & & \Sigma H\mathbf{Z}/2 & = & \Sigma H\mathbf{Z}/2 \\
 & & \uparrow{\scriptstyle w_1} & & \uparrow{\scriptstyle \Omega w_2} \\
BU & \longrightarrow & BO & \longrightarrow & SO \\
\| & & \uparrow & & \uparrow \\
BU & \longrightarrow & BSO & \longrightarrow & Spin
\end{array}$$

The latter fibre sequence shows that $[BG, Spin] = \widehat{JSO}(G)/\widehat{JU}(G)$ since, again, $[BG, B^2U] = [BG, SU] = 0$.

The Bockstein spectral sequence

The cofiber sequence

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \xrightarrow{R} \Sigma^2 ko$$

results in a Bockstein spectral sequence

$$ku^*(X)[\eta] \Longrightarrow ko^*(X)$$

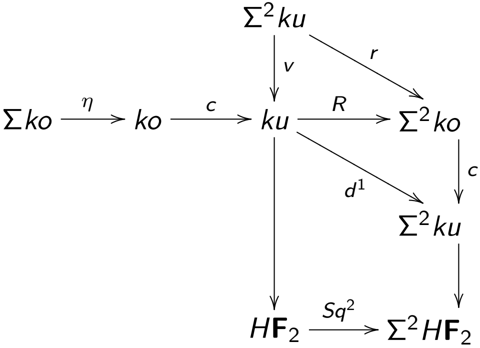
with first differential cR

- closely related to $1 + \tau$ in the periodic theory,
- and to Sq^2 in cohomology.

Better, since $\eta^3 = 0$,

- it collapses: $E^4 = E^\infty$
- at E^∞ it is concentrated on lines 0, 1 and 2.

The η-c-R sequence, its differential, and related operations



Adams Spectral Sequence

Since $H^*ku = \mathcal{A}//E(1)$ and $H^*ko = \mathcal{A}//\mathcal{A}(1)$, we have Adams spectral sequences

$$\mathrm{Ext}^{s,t}_{E(1)}(\mathbf{F}_2, H^*BG) \Longrightarrow ku^{-t+s}BG$$

and

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\mathbf{F}_2, H^*BG) \Longrightarrow ko^{-t+s}BG$$

'Accounting device', drastically simplifies multiplicative information.

Typical uses:

- show that we have enough elements to generate,
- show that the Bott map acts monomorphically in a range sufficiently far beyond the edge of periodicity that relations can be accurately detected in periodic K-theory
- verify that the implications of these relations suffice.

End of Part One