THE UNTWISTING ISOMORPHISM

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Mike, here is the motivating example: let X be a G-set and H a subgroup of G. Then we can restrict X to H and induce it back up, getting $G \times_H X$ with left G action, or we can take the product of the G-sets G/H and X with the diagonal G action. There is an evident isomorphism of G-sets

$$\theta : G \times_H X \longrightarrow G/H \times X$$

given by $\theta([g, x]) = (gH, gx)$. That this is well defined is easy:

$$\theta([gh, x]) = (ghH, ghx) = (gH, ghx) = \theta([g, hx]).$$

Clearly. also the map $\theta^{-1}(gH, x) = [g, g^{-1}x]$ satisfies

$$\theta^{-1}(ghH, x) = [gh, h^{-1}g^{-1}x] = [g, x],$$

so is well-defined. That they are G-maps and inverse to one another are easy calculations.

Here is what you need in your thesis.

For a Hopf algebra A over k a, a sub-Hopf algebra B, and a A-module M, we have the same result: there is an isomorphism of A-modules

$$\theta : A \otimes_B M \longrightarrow (A \otimes_B k) \otimes M$$

given by

$$\theta(a \otimes m) = \sum (a' \otimes 1) \otimes a''m$$

where $\psi(a) = \sum a' \otimes a''$. This is well defined:

$$\begin{aligned}
\theta(ab \otimes m) &= \sum (a'b' \otimes 1) \otimes a''b''m \\
&= \sum (a' \otimes \epsilon(b')) \otimes a''b''m \\
&= \sum \epsilon(b')(a' \otimes 1) \otimes a''b''m \\
&= \sum (a' \otimes 1) \otimes a''(\sum \epsilon(b')b'')m \\
&= \sum (a' \otimes 1) \otimes a''bm \\
&= \theta(a \otimes bm).
\end{aligned}$$

Here we use that

$$\psi(ab) = \psi(a)\psi(b) = \left(\sum a' \otimes a''\right)\left(\sum b' \otimes b''\right) = \sum a'b' \otimes a''b'',$$

the fact that $\epsilon(b') \in k$ is central, and the counital identity $\sum \epsilon(b')b'' = b = \sum b'\epsilon(b'')$.

Similarly,

$$\theta^{-1} : (A \otimes_B k) \otimes M \longrightarrow A \otimes_B M$$

given by

$$\theta^{-1}((a \otimes 1) \otimes m) = \sum a' \otimes \chi(a'')m$$

where χ is the antipode of A. This is well defined:

$$\begin{aligned}
\theta^{-1}((ab \otimes 1) \otimes m) &= \sum a'b' \otimes \chi(b'')\chi(a'')m \\
&= \sum a' \otimes (\sum b'\chi(b''))\chi(a'')m \\
&= \sum a' \otimes \epsilon(b)\chi(a'')m \\
&= \sum a'\epsilon(b) \otimes \chi(a'')m \\
&= \theta^{-1}((a \otimes \epsilon(b)) \otimes m)
\end{aligned}$$

Now, θ is an A-homomorphism:

$$\begin{aligned}
\theta(a_1(a \otimes m)) &= \theta(a_1 a \otimes m) \\
&= \sum (a_1' a' \otimes 1) \otimes a_1'' a'' m \\
&= a_1 \sum (a' \otimes 1) \otimes a'' m \\
&= a_1 \theta(a \otimes m).
\end{aligned}$$

Next, θ^{-1} is an A-homomorphism:

$$\begin{aligned}
\theta^{-1}\,(a_1((a \otimes 1) \otimes m)) &= \theta^{-1}(\sum (a_1' a \otimes 1) \otimes a_1'' m) \\
&= \sum ((a_1')' a' \otimes \chi(a'')\chi((a_1')'')a_1'' m \\
&= \sum (a_1' a' \otimes \chi(a'')(\sum \chi((a_1'')')(a_1'')'')m \\
&= \sum (a_1' a' \otimes \chi(a'')\epsilon(a_1'')m \\
&= \sum \left(\sum a_1'\epsilon(a_1'')\right) a' \otimes \chi(a'')m \\
&= \sum a_1 a' \otimes \chi(a'')m \\
&= a_1 \sum a' \otimes \chi(a'')m \\
&= a_1 \theta^{-1}(a \otimes m).
\end{aligned}$$

Now,

$$\begin{aligned}
\theta^{-1}(\theta(a \otimes m)) &= \theta^{-1}\left(\sum (a' \otimes 1) \otimes a'' m\right) \\
&= \sum (a')' \otimes \chi((a')'')a'' m \\
&= \sum a' \otimes \chi((a'')')(a'')'' m \\
&= \sum a' \otimes \left(\sum \chi((a'')')(a'')''\right) m \\
&= \sum a' \otimes \epsilon(a'')m \\
&= \sum a'\epsilon(a'') \otimes m \\
&= a \otimes m
\end{aligned}$$

so that $\theta^{-1}\theta$ is the identity.

Finally,

$$\begin{aligned}
\theta\left(\theta^{-1}((a \otimes 1) \otimes m)\right) &= \theta\left(\sum a' \otimes \chi(a'')m\right) \\
&= \sum ((a')' \otimes 1) \otimes (a')'' \chi(a'')m \\
&= \sum (a' \otimes 1) \otimes \left(\sum (a'')' \chi(a'')''\right) m \\
&= \sum (a' \otimes 1) \otimes \epsilon(a'')m \\
&= \sum \left(\sum a' \epsilon(a'') \otimes 1\right) \otimes m \\
&= (a \otimes 1) \otimes m,
\end{aligned}$$

so that $\theta\theta^{-1}$ is the identity.

Note that we have several times used without mention the fact that coassociativity, $(\psi \otimes 1)\psi = (1 \otimes \psi)\psi$, says that

$$\sum (a')' \otimes (a')'' \otimes a'' = \sum a' \otimes (a'')' \otimes (a'')''.$$