Radicals and Torsion Theories in Locally Compact Abelian Groups

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Abstract

In this paper we will study the properties of locally compact Abelian Hausdorff topological groups (hereafter known as LCA groups) by means of their mapping properties. The results contained herein are an outgrowth of work done by Professor Armacost [A1] on "sufficiency classes" of LCA groups. The sufficiency class $\underline{S}(H)$ of an LCA group H is the class of all LCA groups G such that there are sufficiently many continuous homomorphisms from G to H to separate the points of G. This condition is easily seen to be equivalent to the requirement that $\bigcap \ker(f)=0$, where f ranges over all elements of (G,H), the set of continuous homomorphisms from G to H. This suggests consideration of the subgroup $R_H(G)=\bigcap_{f\in(G,H)} \ker(f)$ in any LCA group G. Then $\underline{S}(H)$ is just the class of groups G such that $R_H(G)=0$. The subgroups $R_H(G)$ are canonical in the following sense: if $a:G_1\rightarrow G_2$ is a continuous homomorphism, then $a(R_H(G_1))$ is contained in $R_H(G_2)$. This means that R_H can be considered a subfunctor of the identity functor on $\mathcal{L}$, the category consisting of LCA groups as objects and continuous homomorphisms as morphisms. The obvious generalization then is to consider arbitrary subfunctors of the identity on $\mathcal{L}$. Now we cannot say very much about something this general. However, given certain natural restrictions we can prove quite a lot. Namely, if we assume that the

subfunctor of the identity $r:\mathcal{L}\to\mathcal{L}$ is idempotent $(r(r(G))= r(G)$ for all LCA groups G) and a radical $(r(G/r(G))=0$ for all LCA groups G) then it can be shown that $r(G)=\bigcap_{\substack{f\in(G,H)\\ H\in \underline{C}}}\ker(f)$ for a well defined class $\underline{C}$ of LCA groups. Dualizing the above considerations we can also show that any idempotent radical is of the form $r(G)=\overline{\sum_{\substack{f\in(H,G)\\ H\in \underline{C}}}\operatorname{im}(f)}$, so that every idempotent radical can be explicitly constructed in two essentially distinct ways.

Returning to our remarks about the sufficiency class $\underline{S}(H)$ in connection with $R_H(G)$, we note that $\underline{S}(H)$ is only one of two classes of LCA groups distinguished by R_H. The other is the class of all LCA groups G such that $R_H(G)=G$. Obviously this condition is equivalent to the assertion that there are no nontrivial continuous homomorphisms from G to H. Let us denote this latter class by $\underline{T}(H)$. The pair $(\underline{T}(H),\underline{S}(H))$ has some interesting properties. There are no nontrivial homomorphisms from a member of $\underline{T}(H)$ to a member of $\underline{S}(H)$, and $\underline{T}(H)$ is maximal with respect to this property. If R_H is idempotent then $\underline{S}(H)$ is also maximal with respect to this property, and conversely. Abstracting, we define a torsion theory for LCA groups to be a pair $(\underline{T},\underline{F})$ of classes of LCA groups such that there are no continuous homomorphisms from a member of $\underline{T}$ to a member of $\underline{F}$ and such that $\underline{T}$ and $\underline{F}$ are maximal with respect to this property. Examples of torsion theories abound: T contains all connected groups and F contains all totally disconnected groups, or T contains all densely divisible groups and F contains all reduced groups, or T contains all groups with every element compact and F contains all groups with no compact elements. In fact every idempotent radical gives rise to a torsion theory. Now, a torsion theory also yields in a natural way an idempotent radical and, remarkably enough, these correspondences are inverse to each other. Thus we have a 1-1 correspondence between torsion theories and idempotent radicals. This correspondence and the representation of any idempotent radical as the intersection of kernels of continuous homomorphisms and as the closure of the sum of images of continuous homomorphisms are the primary results of the first section.

In the second section we devote our attention to specific radicals, sufficiency classes, torsion theories and their duals. In so doing we characterize many important classes of LCA groups by their mapping properties. We also characterize several important canonical subgroups of LCA groups, some of which, such as the component of the identity and the subgroup of all compact elements, are well known, others of which are not well known but have sufficiently important properties to be worthy of attention. It is the author's hope that these investigations will prove helpful in elucidating the structure of LCA groups. In the final section we turn our attention to problems of this nature. This section is the least complete and the most open-ended. We attempt to say as much as can be said at present about these problems and to indicate possible approaches and plausible conjectures, at least a few of which we hope will be proved at some point in the future.

Terminology and Conventions

All groups will be assumed to be Abelian. The word "group" will mean LCA group, that is, locally compact Abelian Hausdorff topological group. A group without topology will be called a discrete group. A capital roman letter denotes a group unless otherwise mentioned. We will write the group operation additively with one exception. The circle group, T, will be written multiplicatively.

If H is a subset of a group then $\overline{H}$ denotes the closure of H. We write "$H \leq G$" to indicate that H is a closed subgroup of G. Topological isomorphism is denoted by "$\approx$". The symbol "0" is used in various contexts to denote the integer 0, the identity element of a group, the subgroup containing only the identity element and the zero homomorphism.

$\prod_{i \in I} G_i$ is the direct product of the groups G_i with the product topology. If there are a finite number of factors, we may write $G_1 \times G_2 \times \cdots \times G_n$ for the direct product. If all of the G_i's are topologically isomorphic to G we will write G^M for the product where M is the cardinality of the index set. $\bigoplus_{i \in I} G_i$ is the direct sum of the groups G_i. If M is a cardinal number, $\bigoplus_M G$ denotes the direct sum of M copies of G.

The terms "map", "mapping", and "continuous homomorphism" will be used interchangeably for the sake of euphony. We use "(G,H)" to denote the (usually discrete) group of continuous homomorphisms from G to H under the pointwise operation. The identity map on a group G will be denoted by "1_G" or, if there is no chance of confusion, by "1". The notation "$f:G\to H$" denotes a map f from G to H. If a name is not needed for the map we write "$G\to H$". A phrase such as " there exists nonzero $G\to H$" should be read as "there exists a nonzero map from G to H". We use "ker(f)" and "im(f)" to denote the kernel and the image, respectively, of a map f. If $f:G\to H$ and $h:H\to K$ then $hf:G\to K$ is the composition, $hf(x)=h(f(x))$ for all $x\in G$. If $f:G_1\to G_2$ and $H\leq G_1$ then the restriction of f to H is denoted $f|_H$. If $H\leq G$, the natural map $H\to G$ is the inclusion map. The natural map $G\to G/H$ is the map that sends x to the coset x+H.

If G is an LCA group then "$\hat{G}$" and "$(G)^\wedge$" denote the character group of G. If $H\subset G$ then "$A(\hat{G},H)$" denotes the annihilator of H in $\hat{G}$, that is $\{f\in\hat{G}: f(H)=\{1\}\}$. If $a:G\to H$ then $\hat{a}:\hat{H}\to\hat{G}$ is the dual map defined by $\hat{a}(f)=fa\in\hat{G}$ for $f\in\hat{H}$.

$\mathcal{L}$ is the category which has LCA groups as objects and continuous homomorphisms as morphisms.

We say that a class $\underline{C}$ of LCA groups is closed under closed subgroups if $H\leq G$ and $G\in\underline{C}$ imply that $H\in\underline{C}$. The class $\underline{C}$ is closed under quotients if $H\leq G$ and $G\in\underline{C}$ imply that $G/H\in\underline{C}$. It is closed under extensions if $H\in\underline{C}$ and $G/H\in\underline{C}$ imply $G\in\underline{C}$. Classes of LCA groups are generally denoted by underlined capital roman letters.

The groups which we use most frequently are listed below. Complete descriptions may be found in [H&R] or [F].

Q -rationals taken (discrete)

$\mathbb{R}$ -reals with usual topology

Z -integers taken discrete

Q/Z -rationals mod 1 (discrete) ($Q/Z = \oplus Z(p^\infty)$)

$Z(n)$ -cyclic group of order n (discrete)

$Z(p^\infty)$ -quasicyclic group, p prime (discrete)

T -circle group $\mathbb{R}/Z$ (compact)

$\hat{Q}$ -dual group of Q (compact)

J_p -p-adic integers, p prime (compact)

F_p -p-adic numbers, p prime

J - $\prod J_p$ over all primes, i.e., $J = J_2 \times J_3 \times J_5 \times \cdots$

Frequently Used Definitions and Results

If $H \leq G$, $A(G,A(\hat{G},H)) = H$, $\hat{H} \approx \hat{G}/A(\hat{G},H)$, $\widehat{G/H} \approx A(\hat{G},H)$

If $H \leq K \leq G$ then $A(\hat{G},K) \leq A(\hat{G},H)$.

$A(\hat{G},0) = \hat{G}$, and $A(\hat{G},G) = 0$.

An element is compact iff the closure of the subgroup generated by that element is compact. The set of compact elements is a closed subgroup.

The component of the identity is a closed subgroup.

A densely divisible group is one which contains a dense divisible subgroup.

Preradicals and Torsion Theories

1.1 General Preradicals

In this section we present some basic definitions and results concerning preradicals which we will need. First, we remark that a subfunctor of the identity on the category $\mathcal{L}$ is a mapping $F:\mathcal{L}\to\mathcal{L}$ which associates with every LCA group A, a closed subgroup F(A) and with every map $a:A\to B$ the restriction map $a|_{F(A)}:F(A)\to F(B)$. Note that $F(a)=a|_{F(A)}$ is well defined if and only if $a(F(A))\leq F(B)$. Now we present some basic definitions.

1. Definition: A preradical is a subfunctor of the identity. A preradical r is idempotent if $r(r(A))=r(A)$ for all LCA groups A. A preradical r is radical if $r(A/r(A))=0$ for all LCA groups A. We will call a radical preradical a radical.

There is a natural order which we can put on preradicals. Namely, if r and s are preradicals, we say that r is less than s (written $r\leq s$) iff r(G) is contained in s(G) for all LCA groups G. We will make use of the order later.

Now, as with virtually every other concept in LCA groups, preradicals occur in dual pairs. Let r be a preradical. We define the dual preradical $\hat{r}$ by:

$$\hat{r}(G)=A(G,r(\hat{G})).$$

Note that if $a:G\to H$ then $\hat{r}(a)$ is $a|_{\hat{r}(G)}$. In the future we will generally assume this without explicit mention. Note also that $r(G)=A(G,\hat{r}(\hat{G}))$. Thus we have $r\approx\hat{\hat{r}}$. Now we must verify the fact that $\hat{r}$ is a preradical. We also note a few simple relations between r and $\hat{r}$.

2. <u>Lemma</u>: (a) $\hat{r}$ is a preradical.

(b) $(\hat{r}(G))\hat{}\approx\hat{G}/r(\hat{G})$ and, dually, $(G/\hat{r}(G))\hat{}\approx r(\hat{G})$.

(c) $\hat{r}$ is idempotent iff r is radical. $\hat{r}$ is radical iff r is idempotent.

<u>Proof</u>: (a) Clearly $\hat{r}(G)\leq G$ so we need only show that if $a:G\to H$ then $a(\hat{r}(G))\leq\hat{r}(H)$. Now, letting $\hat{a}:\hat{H}\to\hat{G}$ be the map dual to a, we have by [H&R, 24.39]

$$A(\hat{H},a(\hat{r}(G)))=\hat{a}^{-1}[A(\hat{G},\hat{r}(G))\cap\hat{a}(\hat{H})]=\hat{a}^{-1}(r(\hat{G})\cap\hat{a}(\hat{H})).$$

Since r is a preradical, $\hat{a}(r(\hat{H}))\leq r(\hat{G})\cap\hat{a}(\hat{H})$. Therefore $r(\hat{H})\leq\hat{a}^{-1}(r(\hat{G})\cap\hat{a}(\hat{H}))$ which implies by the above equality that $r(\hat{H})\leq A(\hat{H},a(\hat{r}(G)))$. Now taking annihilators of each side reverses the containment, so $a(\hat{r}(G))\leq A(H,r(\hat{H}))=\hat{r}(H)$, the last equality following from the definition of $\hat{r}$.

(b) The first statement follows immediately from the definition of $\hat{r}$ and the fact that if $H\leq G$ then $\hat{H}\approx\hat{G}/A(\hat{G},H)$. The second statement is obtained from the first by duality.

(c) Suppose that r is radical. Then

$$\begin{aligned}\hat{r}(\hat{r}(G))&=A(\hat{r}(G),r(\widehat{\hat{r}(G)}))\\&=A(\hat{r}(G),r(\hat{G}/r(\hat{G}))) &&\text{(by (b))}\\&=A(\hat{r}(G),0) &&\text{(since r is radical)}\\&=\hat{r}(G)\end{aligned}$$

proving that $\hat{r}$ is idempotent. Conversely, suppose r is idempotent. Then

$$\hat{r}(G/\hat{r}(G)) = A(G/\hat{r}(G), r(\widehat{G/\hat{r}(G)}))$$
$$= A(\widehat{r(\hat{G})}, r(r(\hat{G}))) \qquad \text{(by (b))}$$
$$= A(\widehat{r(\hat{G})}, r(\hat{G})) \qquad \text{(since r is idempotent)}$$
$$= 0$$

and hence $\hat{r}$ is radical. The other two statements follow by duality.//

Note that if $r \leq s$ then $\hat{s} \leq \hat{r}$.

Now we present a result concerning the behaviour of preradicals with respect to quotients and subgroups.

3. <u>Lemma</u>: (a) If r is a preradical and $K \leq G$ then $r(G/K)=0$ implies $r(G) \leq K$.

(b) If r is a radical and $L \leq r(M)$ then $r(M/L) \leq r(M)/L$.

(c) If r is a preradical and $K \leq G$ then $r(K)=K$ implies that $K \leq r(G)$.

<u>Proof</u>: (a) The restriction of the natural map $a: G \to G/K$ sends $r(G)$ to $r(G/K)=0$. Hence $r(G) \leq \ker(a)=K$.

(b) (This proof is due to Stenström [S]) Applying r to the natural map $M \to M/L$ yields a map $r(M) \to r(M/L)$ which clearly has as kernel $L \cap r(M)=L$. Thus $r(M)/L \leq r(M/L)$. Also, the natural map $M/L \to M/r(M)$ (which exists since $L \leq r(M)$) induces the map $r(M/L) \to r(M/r(M))=0$ since r is radical. Since the kernel of the map $M/L \to M/r(M)$ is $r(M)/L$ this implies that $r(M/L) \leq r(M)/L$, thereby forcing equality.

(c) This result is the dual of (a). Applying r to the natural map $K \to G$ yields an inclusion map $r(K) \to r(G)$. Since $r(K)=K$ this implies that $K \leq r(G)$.//

We now isolate two important classes of LCA groups associated with each preradical in the following

4. Definition: If r is a preradical, let

$\underline{T}_r = \{G: r(G)=G\}$ and $\underline{F}_r = \{G: r(G)=0\}$.

This notation will be used throughout although the preradical may be denoted by a letter other than r. We make one notational convention at this point. If $\underline{C}$ is a class of LCA groups then $\widehat{\underline{C}}$ will denote the class $\{G: \widehat{G} \in \underline{C}\}$.

Notice that it follows immediately from the definitions that r is radical iff $G/r(G) \in \underline{F}_r$ for all G, and similarly r is idempotent iff $r(G) \in \underline{T}_r$ for all G.

In the next lemma we prove some elementary properties of the classes $\underline{T}_r$ and $\underline{F}_r$.

5. Lemma: (a) $\underline{T}_r$ is closed under quotients, that is, if $G \in \underline{T}_r$ and $H \leq G$ then $G/H \in \underline{T}_r$.

(b) $\underline{F}_r$ is closed under closed subgroups, that is, if $G \in \underline{F}_r$ and $H \leq G$ then $H \in \underline{F}_r$.

(c) If $G \in \underline{T}_r$ and $H \in \underline{F}_r$ then $(G,H)=0$.

(d) $\underline{T}_{\hat{r}} = \widehat{\underline{F}_r}$ and $\underline{F}_{\hat{r}} = \widehat{\underline{T}_r}$.

Proof: (a) If $G \in \underline{T}_r$, that is, $r(G)=G$, and $p: G \to G/H$ is the natural map then $G/H = p(G) = p(r(G)) \leq r(G/H)$, and thus $r(G/H) = G/H$ and so $G/H \in \underline{T}_r$.

(b) If $H \leq G$ and $G \in \underline{F}_r$ then the natural map $i: H \to G$ induces $r(H) \to r(G) = 0$. Thus $r(H) \leq \ker(i) = 0$, and therefore $H \in \underline{F}_r$.

(c) Let $G \in \underline{T}_r$ and $H \in \underline{F}_r$. Suppose $a: G \to H$. Then $a(G) = a(r(G)) \leq r(H) = 0$ and hence $a = 0$. Thus $(G,H) = 0$.

(d) Now $G \in \underline{T}_{\hat{r}}$ iff $G = \hat{r}(G) = A(G, r(\hat{G}))$. But $G = A(G, r(\hat{G}))$ iff $r(\hat{G}) = 0$ which is equivalent to the condition that $\hat{G} \in \underline{F}_r$. By definition, $\hat{G} \in \underline{F}_r$ iff $G \in \widehat{\underline{F}_r}$. The other assertion follows by duality.//

The final result in this section relates properties of a preradical r to the classes $\underline{T}_r$ and $\underline{F}_r$ defined by it. We will make frequent use of this result. First, we remark that a class $\underline{C}$ is said to be closed under extensions if whenever $H \leq G$ and $H, G/H \in \underline{C}$ then it follows that $G \in \underline{C}$. Now we state the

6. <u>Lemma</u>: (a) Let r be a radical. Then r is idempotent iff $\underline{F}_r$ is closed under extensions.

(b) Let r be an idempotent preradical. Then r is radical iff $\underline{T}_r$ is closed under extensions.

<u>Proof</u>: (a) Let r be a radical. Assume r is idempotent and let $H \leq G$ such that H and G/H are in $\underline{F}_r$. Since $G/H \in \underline{F}_r$ $r(G/H) = 0$ from which it follows by Lemma 3(a) that $r(G) \leq H$. Therefore $r(r(G)) \leq r(H)$. Since r is idempotent

and since $H \in \underline{F}_r$ this implies that $r(G) \leq r(H) = 0$ from which it follows that $G \in \underline{F}_r$.

Now assume that $\underline{F}_r$ is closed under extensions. Since r is radical, $G/r(G)$ and $r(G)/r(r(G))$ are in $\underline{F}_r$. Now $G/r(G) \approx [G/r(r(G))]/[r(G)/r(r(G))]$ so $G/r(r(G)) \in \underline{F}_r$ since $\underline{F}_r$ is closed under extensions. By Lemma 3(a) we have $r(G) \leq r(r(G))$. But $r(r(G))$ is always contained in $r(G)$ and hence $r(G) = r(r(G))$, proving that r is idempotent.

(b) By Lemma 2(c) and Lemma 5(d), part (b) follows from (a) and the observation that a class $\underline{C}$ is closed under extensions iff $\hat{\underline{C}}$ is closed under extensions. (Suppose H and G/H in $\underline{C}$ implies that G is in $\underline{C}$. Let L and M/L be in $\hat{\underline{C}}$. Then $\hat{L} \approx \hat{M}/A(\hat{M},L)$ and $\widehat{M/L} \approx A(\hat{M},L)$ are in $\underline{C}$ so $\hat{M}$ is in $\underline{C}$. Thus M is in $\hat{\underline{C}}$ and so $\hat{\underline{C}}$ is closed under extensions. Since $\underline{C} = \hat{\hat{\underline{C}}}$, this suffices to prove both directions.)//

1.2 Torsion Theories

We now define torsion theories and prove some basic results about them which we will need.

7. Definition: A torsion theory for $\mathcal{L}$ is a pair $(\underline{T},\underline{F})$ of classes of LCA groups such that

(i) $(G,H) = 0$ for all $G \in \underline{T}$ and $H \in \underline{F}$, and

(ii) $\underline{T}$ and $\underline{F}$ are maximal with respect to (i).

$\underline{T}$ is called a torsion class, $\underline{F}$ a torsion-free class.

Notice that $(\underline{T},\underline{F})$ is a torsion theory iff $(\hat{\underline{F}},\hat{\underline{T}})$ is a torsion theory. This follows immediately from the fact that $(G,H) = 0$ iff $(H,G) = 0$ where G and H are LCA groups [H&R, 24.41.a].

In the next lemma we present some necessary conditions for a pair $(\underline{T},\underline{F})$ to be a torsion theory. We do not know whether these conditions are also sufficient. In fact, we know of no intrinsic characterization of torsion classes or torsion-free classes.

8. Lemma: If $(\underline{T},\underline{F})$ is a torsion theory then

(a) $\underline{T}$ is closed under quotients, extensions and direct sums when defined.

(b) $\underline{F}$ is closed under closed subgroups, extensions and direct products when defined.

Proof: (a) It is clear that $(G/H,F) \neq 0$ implies $(G,F) \neq 0$. Hence $\underline{T}$ is closed under quotients.

Suppose that $H \leq G$ such that H and G/H are in $\underline{T}$. Let $a \in (G,F)$ for some $F \in \underline{F}$. Now $a|_H \in (H,F)$ so $a|_H = 0$. Thus

$H \leq \ker(a)$ and so the definition $a^*(g+H) = a(g)$ for all $g \in G$ gives a well defined element a^* of $(G/H,F)$. But $(G/H,F) = 0$ and hence $a^* = 0$, which implies that $a = 0$. Since a was an arbitrary element of (G,F) it follows that $(G,F) = 0$ and thus, by maximality of $\underline{T}$, $G \in \underline{T}$. Hence $\underline{T}$ is closed under extensions.

Now let $G_i \in \underline{T}$ for each i in some index set I, such that the direct sum $\oplus G_i$ is an LCA group (topologized in such a way that $\widehat{\oplus G_i} \approx \prod \widehat{G}_i$ - it is known that $\oplus G_i$ is an LCA group iff all but a finite number of the G_i's are discrete [FGI, remark following Theorem 2.12]). If $a: \oplus G_i \rightarrow F$ with $F \in \underline{F}$ and $a \neq 0$, then $a|_{G_i} \neq 0$ for some i. But this is impossible since each $G_i \in \underline{T}$. Thus $(\oplus G_i, F) = 0$ and $\oplus G_i \in \underline{T}$ since F was an arbitrary member of $\underline{F}$. Hence $\underline{T}$ is closed under direct sums when they are defined.

(b) If $H \leq G$ then clearly $(T,H) \neq 0$ implies that $(T,G) \neq 0$. Hence $\underline{F}$ is closed under closed subgroups. (Although it is not apparent from the proof, the restriction to <u>closed</u> subgroups is necessary since a non-closed subgroup of a locally compact group is not locally compact [H&R, 5.11] and we are dealing only with classes of LCA groups.)

We prove that $\underline{F}$ is closed under extensions by duality. Since $(\underline{T},\underline{F})$ is a torsion theory, $(\hat{\underline{F}},\hat{\underline{T}})$ is also a torsion theory by the remark preceding this lemma. By part (a) of the lemma, $\hat{\underline{F}}$ is closed under extensions. As shown in the proof of Lemma 6(b), this implies $\underline{F}$ is closed under extensions.

Now let $G_i \in \underline{F}$ for each i in some index set I, such that the direct product $\prod G_i$ is an LCA group under the product topology. (It is known that $\prod G_i$ is an LCA group iff all but a finite number of the G_i's are compact [H&R, 6.4].) Let $p_j : \prod_i G_i \to G_j$ be the projection mapping. Now if $a: T \to \prod G_i$ with $a \neq 0$ for some $T \in \underline{T}$ then $p_j a \neq 0$ for some j. But this is impossible since $(T, G_j) = 0$ for all j. Thus $a = 0$ and so $(T, \prod G_i) = 0$. Since T was arbitrary, this proves, by the maximality of $\underline{F}$, that $\prod G_i \in \underline{F}$.//

Note that we could have proved that $\underline{F}$ is closed under extensions directly. The proof by duality is shorter. Also note that we could have proved by duality that $\underline{F}$ is closed under direct products when they are defined by using the fact that $\oplus G_i$ is defined exactly when $\prod \widehat{G}_i$ is defined and $\widehat{\oplus G_i} \approx \prod \widehat{G}_i$.

In our next definition we show how to produce a torsion theory in two dual ways from a given class of LCA groups.

9. <u>Definition</u>: Given a class $\underline{C}$ of LCA groups, let

$\underline{F} = \{F \in \mathcal{L} : (C,F) = 0 \text{ for all } C \in \underline{C}\}$ and let

$\underline{T} = \{T \in \mathcal{L} : (T,F) = 0 \text{ for all } F \in \underline{F}\}$.

It is immediate that $(\underline{T}, \underline{F})$ is a torsion theory. We call $(\underline{T}, \underline{F})$ the <u>torsion theory generated by</u> $\underline{C}$. If we let

$\underline{T} = \{T \in \mathcal{L} : (T,C) = 0 \text{ for all } C \in \underline{C}\}$ and let

$\underline{F} = \{F \in \mathcal{L}: (T,F) = 0 \text{ for all } T \in \underline{T}\}$

then ($\underline{T}$,$\underline{F}$) is the torsion theory cogenerated by $\underline{C}$.

We state the following result as a lemma in order to facilitate reference to it.

10. Lemma: (a) If ($\underline{T}$,$\underline{F}$) is the torsion theory generated by $\underline{C}$ then $\underline{T}$ is the smallest torsion class containing $\underline{C}$.

(b) If ($\underline{T}$,$\underline{F}$) is the torsion theory cogenerated by $\underline{C}$ then $\underline{F}$ is the smallest torsion-free class containing $\underline{C}$.

Proof: (a) Suppose ($\underline{T}_1$,$\underline{F}_1$) is a torsion theory with $\underline{C} \subset \underline{T}_1$. Then clearly $(C,F) = 0$ for all $C \in \underline{C}$ and $F \in \underline{F}_1$. Thus $\underline{F}_1 \subset \underline{F}$ by the definition of $\underline{F}$. Then $(T,F) = 0$ for all $T \in \underline{T}$ and $F \in \underline{F}_1$ from which it follows by the maximality of $\underline{T}_1$ that $\underline{T} \subset \underline{T}_1$.

(b) Dually, if ($\underline{T}_1$,$\underline{F}_1$) is a torsion theory with $\underline{C} \subset \underline{F}_1$, it follows that $\underline{T}_1 \subset \underline{T}$ and hence $\underline{F} \subset \underline{F}_1$.//

We now turn our attention to the classes $\underline{T}_r$ and $\underline{F}_r$ defined in the previous section in order to prove our first result connecting preradicals and torsion theories.

11. Lemma: If r is an idempotent radical then ($\underline{T}_r$,$\underline{F}_r$) is a torsion theory.

Proof: By Lemma 5(c), $(T,F) = 0$ for all $T \in \underline{T}_r$ and $F \in \underline{F}_r$. Now, suppose $G \notin \underline{T}_r$. Then, by definition, $r(G) \neq G$ and hence $G/r(G) \neq 0$. Thus, $(G,G/r(G)) \neq 0$ since it contains the natural map $G \to G/r(G)$. Now $G/r(G) \in \underline{F}_r$ since r is radical, proving that $\underline{T}_r$ is maximal with respect to the requirement that $(T,F) = 0$ for all $T \in \underline{T}_r$ and $F \in \underline{F}_r$. Dually, suppose $G \notin \underline{F}_r$. Then $r(G) \neq 0$ and since r is idempotent $r(G) \in \underline{T}_r$. Thus $(r(G),G) \neq 0$, proving that $\underline{F}_r$ is also maximal. Hence, $(\underline{T}_r, \underline{F}_r)$ is a torsion theory.//

Note that if r is either idempotent or radical and if $(\underline{T}_r, \underline{F}_r)$ is a torsion theory, then r is both idempotent and radical by Lemmas 6 and 8.

1.3 Kernel Radicals

We now turn our attention to a particular method of constructing radicals from a given class of groups. We start with a

12. Definition: Let $\underline{C}$ be a class of LCA groups. The $\underline{C}$-kernel radical $r_{\underline{C}}$ is defined by $r_{\underline{C}}(G)=\bigcap_{\substack{f\in(G,C)\\ C\in\underline{C}}}\ker(f)$.

In the following lemma we show that we are justified in calling $r_{\underline{C}}$ a radical. We also show the close connection between $r_{\underline{C}}$ and the torsion theory cogenerated by $\underline{C}$.

13. Lemma: Let $r=r_{\underline{C}}$, the $\underline{C}$-kernel radical. Let $(\underline{T}_r,\underline{F}_r)$ be as defined in Definition 4, and let $(\underline{T}_0,\underline{F}_0)$ be the torsion theory cogenerated by $\underline{C}$. Then

(a) r is a radical

(b) $\underline{T}_0=\underline{T}_r$ and $\underline{F}_r\subset\underline{F}_0$

(c) The following are equivalent: (i) r is idempotent, (ii) $(\underline{T}_r,\underline{F}_r)$ is a torsion theory, and (iii) $\underline{F}_r=\underline{F}_0$.

Proof: (a) First we must show that r is a preradical. This amounts to showing that if $a:G\to H$ then $a(r(G))\leq r(H)$. Now if $a(g)\notin r(H)$ for some $g\in G$ then $f(a(g))\neq 0$ for some $f:H\to C$ with $C\in\underline{C}$. But then $fa:G\to C$ and $fa(g)\neq 0$ so $g\notin r(G)$. Hence, $a(r(G))\leq r(H)$.

Now we must show that $r(G/r(G))=0$ for all G.

Suppose $x+r(G) \in G/r(G)$ with $x \notin r(G)$. Then $f(x) \neq 0$ for some $f: G \to C$ with $C \in \underline{C}$. Since $r(G) \leq \ker(f)$ by the definition of $r(G)$, f induces a well defined map $f^*: G/r(G) \to C$ by the rule $f^*(g+r(G)) = f(g)$. Thus $f^*(x+r(G)) \neq 0$ and so $x+r(G) \notin r(G/r(G))$. Since $x+r(G)$ was an arbitrary nonzero element in $G/r(G)$, this proves that $r(G/r(G)) = 0$, and therefore r is a radical.

(b) First we note that $G \in \underline{T}_r$ iff $r(G) = G$ and $G \in \underline{T}_0$ iff $(G,C) = 0$ for all $C \in \underline{C}$. Clearly the nonexistence of a nonzero map from G to a member of $\underline{C}$ is equivalent to the condition that the kernel of every map from G to a member of $\underline{C}$ is G. Thus $\underline{T}_r = \underline{T}_0$. Now by Lemma 5(c) and the fact that $\underline{T}_r = \underline{T}_0$ we have $(T,F) = 0$ for all $T \in \underline{T}_0$ and $F \in \underline{F}_r$. By the maximality of $\underline{F}_0$ with respect to this property, we have $\underline{F}_r \subset \underline{F}_0$.

(c) We prove the following implications which yield the equivalence of the three conditions: (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i). That (i)$\Rightarrow$(ii) follows immediately from (a) and Lemma 11. Now if $(\underline{T}_r, \underline{F}_r)$ is a torsion theory, then, since $\underline{T}_0 = \underline{T}_r$ by (b), we must have $\underline{F}_0 = \underline{F}_r$ since both $\underline{F}_0$ and $\underline{F}_r$ are maximal with respect to the same torsion class. Thus (ii)$\Rightarrow$(iii). If $\underline{F}_0 = \underline{F}_r$ then $(\underline{T}_0, \underline{F}_0) = (\underline{T}_r, \underline{F}_r)$ by (b), from which it follows that $(\underline{T}_r, \underline{F}_r)$ is a torsion theory since $(\underline{T}_0, \underline{F}_0)$ is a torsion theory. Thus (iii)$\Rightarrow$(ii). If $(\underline{T}_r, \underline{F}_r)$ is a torsion theory then, by Lemma 8(b), $\underline{F}_r$ is closed under extensions. Then, by Lemma 6(a) r is idempotent, proving that (ii)$\Rightarrow$(i).//

Our next result is the first major step toward showing the 1-1 correspondence between torsion theories and idempotent radicals. The result states that if $(\underline{T},\underline{F})$ is a torsion theory, the $\underline{F}$-kernel radical, $r_{\underline{F}}$, is idempotent and so $(\underline{T}_{r_{\underline{F}}},\underline{F}_{r_{\underline{F}}})$ (as in Definition 4) is a torsion theory. The result states further that $(\underline{T}_{r_{\underline{F}}},\underline{F}_{r_{\underline{F}}})=(\underline{T},\underline{F})$, the original torsion theory. This has as corollary a result we state in the following section: every torsion theory is of the form $(\underline{T}_r,\underline{F}_r)$ for some idempotent kernel radical r. We also have as an immediate corollary that if $(\underline{T}_1,\underline{F}_1)$ and $(\underline{T}_2,\underline{F}_2)$ are distinct torsion theories then $r_{\underline{F}_1}$ and $r_{\underline{F}_2}$ are distinct idempotent radicals. We now state and prove the

14. <u>Theorem</u>: If $(\underline{T},\underline{F})$ is a torsion theory, r is $r_{\underline{F}}$, the $\underline{F}$-kernel radical, and $(\underline{T}_r,\underline{F}_r)$ is as in Definition 4, then r is an idempotent radical and $(\underline{T},\underline{F})=(\underline{T}_r,\underline{F}_r)$.

<u>Proof</u>: It is clear that $\underline{F}$ cogenerates $(\underline{T},\underline{F})$ by Lemma 10(b). By the previous lemma then, r is a radical, $\underline{T}=\underline{T}_r$, and $\underline{F}_r\subset\underline{F}$. Now, if $G\in\underline{F}$ then $r(G)=\bigcap_{f:G\to C\in\underline{F}}\ker(f)\leq\ker(1_G)=0$. Hence $G\in\underline{F}_r$, implying that $\underline{F}=\underline{F}_r$ and so $(\underline{T},\underline{F})=(\underline{T}_r,\underline{F}_r)$. Thus by Lemma 13(c), r is idempotent.//

1.4 Correspondence Between Idempotent Radicals and Torsion Theories, Part 1

In this section we show that every idempotent radical is a kernel radical for an appropriately chosen class and that there is a 1-1 correspondence between idempotent radicals and torsion theories. First, we define the radicals and classes we will work with, and then prove a lemma which is of interest in itself. The lemma contains an explicit construction of the largest idempotent radical not greater than a given radical.

Let r be a preradical (not necessarily idempotent or radical). Let $(\underline{T}_r,\underline{F}_r)$ be as in Definition 4. Let $(\underline{T}_0,\underline{F}_0)$ be the torsion theory cogenerated by $\underline{F}_r$ and let $r_{\underline{F}_0}$ be the $\underline{F}_0$-kernel radical. As noted before, $\underline{F}_0$ clearly cogenerates $(\underline{T}_0,\underline{F}_0)$. By Theorem 14, $r_{\underline{F}_0}$ is an idempotent radical and $(\underline{T}_0,\underline{F}_0)=(\underline{T}_{r_{\underline{F}_0}},\underline{F}_{r_{\underline{F}_0}})$. Using this notation we state the

15. Lemma: (a) If r is an idempotent radical then $r=r_{\underline{F}_0}$.

(b) If s is an idempotent radical not greater than r then $s\le r_{\underline{F}_0}$. If r is radical then $r_{\underline{F}_0}\le r$.

Proof: (a) Although (b) implies (a), we need (a) to be able to prove (b). If r is an idempotent radical then by Lemma 11, $(\underline{T}_r,\underline{F}_r)$ is a torsion theory. Since $(\underline{T}_0,\underline{F}_0)$ is the torsion theory cogenerated by $\underline{F}_r$, $(\underline{T}_0,\underline{F}_0) = (\underline{T}_r,\underline{F}_r)$ by Lemma 10(b). In particular, if G is an LCA group then $r(G) = 0$ iff $r_{\underline{F}_0}(G) = 0$, using the fact that $\underline{F}_0 = \underline{F}_{r_{\underline{F}_0}}$ by Theorem 14. Now, for any LCA group G, $r(G/r(G)) = 0$ and therefore $r_{\underline{F}_0}(G/r(G)) = 0$. Hence $r_{\underline{F}_0}(G) \leq r(G)$ by Lemma 3(a). Similarly, $r(G) \leq r_{\underline{F}_0}(G)$ from which it follows that $r = r_{\underline{F}_0}$.

(b) Let s be an idempotent radical such that $s(G) \leq r(G)$ for all LCA groups G. Let $(\underline{T}_s,\underline{F}_s)$ be the torsion theory as in Definition 4 and Lemma 11. If $(\underline{T}_{0s},\underline{F}_{0s})$ is the torsion theory cogenerated by $\underline{F}_s$, and $s_{\underline{F}_{0s}}$ is the $\underline{F}_{0s}$-kernel radical then, by Lemma 10(b), $(\underline{T}_s,\underline{F}_s) = (\underline{T}_{0s},\underline{F}_{0s})$ and by part (a) of this lemma, $s = s_{\underline{F}_{0s}}$. Now $\underline{F}_r \subset \underline{F}_s$ since $s(G) \leq r(G)$ for all G, and thus, by Lemma 10(b), $\underline{F}_0 \subset \underline{F}_s$. Putting all these together, we have, for any LCA group G,

$$s(G) = s_{\underline{F}_{0s}}(G) = \bigcap_{\substack{f:G\to F \\ F\in\underline{F}_{0s}=\underline{F}_s}} \ker(f) \leq \bigcap_{\substack{f:G\to F \\ F\in\underline{F}_0}} \ker(f) = r_{\underline{F}_0}(G).$$

Thus $s \leq r_{\underline{F}_0}$. Now if r is radical then $G/r(G) \in \underline{F}_r \subset \underline{F}_0$ for all G. Thus $r_{\underline{F}_0}(G/r(G)) \leq \ker(G/r(G) \xrightarrow{1} G/r(G)) = 0$ for all G. Then, by Lemma 3(a), $r_{\underline{F}_0}(G) \leq r(G)$ for all G, that is, $r_{\underline{F}_0} \leq r$.//

We now come to the main theorem. We include restatements of two results already proved in order to have these three results together.

16. Theorem: (a) Every idempotent radical is a kernel radical.

(b) Every torsion theory has the form $(\underline{T}_r, \underline{F}_r)$ for some idempotent kernel radical r.

(c) There is a 1-1 correspondence between idempotent radicals and torsion theories.

Proof: (a) This is immediate from Lemma 15(a).

(b) This is immediate from Theorem 14.

(c) We associate with each idempotent radical r, the torsion theory $(\underline{T}_r, \underline{F}_r)$ using Lemma 11, and with each torsion theory $(\underline{T}, \underline{F})$ the idempotent $\underline{F}$-kernel radical $r_{\underline{F}}$ using Theorem 14. Now we must show that these correspondences are inverse to each other, that is, we must show that $(\underline{T}, \underline{F}) = (\underline{T}_{r_{\underline{F}}}, \underline{F}_{r_{\underline{F}}})$ and that $r = r_{\underline{F}_r}$. The former was shown in Theorem 14 and the latter follows from Lemma 15(a) and the observation that $\underline{F}_0 = \underline{F}_r$ since r is an idempotent radical.//

1.5 Image Preradicals

In this section we give a method of constructing idempotent preradicals which is dual to the construction of kernel radicals. We prove results dual to the results of section 3. We also prove results about the duality between image preradicals and kernel radicals. We start with a

17. Definition: Let $\underline{C}$ be a class of LCA groups. The $\underline{C}$-image preradical $s_{\underline{C}}$ is defined by $s_{\underline{C}}(G) = \overline{\sum_{\substack{f:C\to G\\ C\in \underline{C}}} \mathrm{im}(f)}$.

Now there is a complete duality between image preradicals and kernel radicals. Specifically, we have the following

18. Lemma: Let s be the $\underline{C}$-image preradical. Let r be the $\hat{\underline{C}}$-kernel radical. Let $(\underline{T}_0, \underline{F}_0)$ be the torsion theory generated by $\underline{C}$. Let $(\underline{T}_1, \underline{F}_1)$ be the torsion theory cogenerated by $\hat{\underline{C}}$. Let $(\underline{T}_s, \underline{F}_s)$ and $(\underline{T}_r, \underline{F}_r)$ be as defined in Definition 4. Then:

(a) $s = \hat{r}$.

(b) s is an idempotent preradical.

(c) $\underline{T}_s = \widehat{\underline{F}_r}$ and $\underline{F}_s = \widehat{\underline{T}_r}$.

(d) $\underline{T}_0 = \widehat{\underline{F}_1}$ and $\underline{F}_0 = \widehat{\underline{T}_1}$.

(e) $\underline{F}_0 = \underline{F}_s$ and $\underline{T}_s \subset \underline{T}_0$.

(f) The following are equivalent: (i) s is radical, (ii) $(\underline{T}_s, \underline{F}_s)$ is a torsion theory, and (iii) $\underline{T}_s = \underline{T}_0$.

Proof: (a) By [A2,4.2] we have $A(\hat{G}, \bigcap_{f:G\to C} \ker(f)) = \overline{\sum_{h:\hat{C}\to\hat{G}} h(\hat{C})}$.

Then, it follows that

$$\hat{r}(G) = A(G, r(\hat{G}))$$

$$= \overline{\sum_{C \in \hat{\underline{C}}} A(G, \bigcap_{f:\hat{G}\to C} \ker(f))} \quad \text{(by [H\&R, 23.29.b and 24.10])}$$

$$= \overline{\sum_{C \in \hat{\underline{C}}} (\overline{\sum_{h:\hat{C}\to G} \operatorname{im}(h)})} \quad \text{(by the result mentioned above)}$$

$$= \overline{\sum_{\substack{h:\hat{C}\to G \\ C \in \hat{\underline{C}}}} \operatorname{im}(h)} \quad \text{(a tedious but elementary calculation)}$$

$$= \overline{\sum_{\substack{h:C\to G \\ C \in \underline{C}}} \operatorname{im}(h)} = s(G).$$

(b) Immediate from (a) and Lemma 2(c).

(c) Immediate from (a) and Lemma 5(d).

(d) This follows immediately from the definitions using the fact that $(G,H) = 0$ iff $(\hat{H},\hat{G}) = 0$ (see the remark following Definition 7).

(e) Using (c), (d) and Lemma 13(b) we have $\underline{F}_0 = \hat{\underline{T}}_1 = \hat{\underline{T}}_r = \underline{F}_s$. Similarly, $\underline{T}_s = \hat{\underline{F}}_r \subset \hat{\underline{F}}_1 = \underline{T}_0$.

(f) By Lemma 2(c), s is radical iff r is idempotent. By (c) and the remark following Definition 7, $(\underline{T}_s,\underline{F}_s)$ is a torsion theory iff $(\underline{T}_r,\underline{F}_r)$ is a torsion theory. Finally, it is clear from (c) and (d) that $\underline{T}_s = \underline{T}_0$ iff $\underline{F}_r = \underline{F}_1$. Thus this result follows by duality from Lemma 13(c).//

We now have a result dual to Theorem 14. It will play the same rôle in section 6 that Theorem 14 played in section 4.

19. Theorem: If $(\underline{T},\underline{F})$ is a torsion theory, s is $s_{\underline{T}}$, the

$\underline{T}$-image preradical, and $(\underline{T}_s,\underline{F}_s)$ is as in Definition 4 then s is an idempotent radical and $(\underline{T},\underline{F}) = (\underline{T}_s,\underline{F}_s)$.

Proof: Clearly the torsion theory generated by $\underline{T}$ is $(\underline{T},\underline{F})$. By Lemma 18, s is an idempotent preradical, $\underline{F}_s = \underline{F}$, and $\underline{T}_s \subset \underline{T}$. Now, if $G \in \underline{T}$ then $s(G) = s_{\underline{T}}(G) \geq \text{im}(G \xrightarrow{1} G) = G$ so that $G \in \underline{T}_s$. Therefore, $(\underline{T},\underline{F}) = (\underline{T}_s,\underline{F}_s)$ and by Lemma 18(f), s is radical.//

Our final result in this section deals with the relationship between image preradicals and kernel radicals.

20. Theorem: There is a 1-1 correspondence between image preradicals and kernel radicals. Furthermore, there is a 1-1 correspondence between image radicals and idempotent kernel radicals.

Proof: The first statement is immediate from Lemma 18(a) and its dual, $\hat{s} = r$, which follows from the fact that $r = \hat{\hat{r}}$ (see the remark preceding Lemma 2). The second statement follows from the first and Lemma 2(c).//

1.6 Correspondence Between Idempotent Radicals and Torsion Theories, Part 2

In this section we prove results dual to those of section 4. Let s be an arbitrary preradical and let $(\underline{T}_s, \underline{F}_s)$ be as in Definition 4. Let $(\underline{T}_0, \underline{F}_0)$ be the torsion theory generated by $\underline{T}_s$, and let $s_{\underline{T}_0}$ be the $\underline{T}_0$-image preradical. By Theorem 19, $s_{\underline{T}_0}$ is an idempotent radical and $(\underline{T}_0, \underline{F}_0) = (\underline{T}_{s_{\underline{T}_0}}, \underline{F}_{s_{\underline{T}_0}})$. Using this notation we have the following

21. Lemma: (a) If s is an idempotent radical then $s = s_{\underline{T}_0}$.

(b) If r is an idempotent radical not less than s then $s_{\underline{T}_0} \leq r$. If s is idempotent then $s \leq s_{\underline{T}_0}$.

Proof: (a) If s is an idempotent radical then by Lemma 11 $(\underline{T}_s, \underline{F}_s)$ is a torsion theory, from which it follows that $(\underline{T}_s, \underline{F}_s) = (\underline{T}_0, \underline{F}_0)$ by Lemma 10(a). Therefore $s(G) = 0$ iff $s_{\underline{T}_0}(G) = 0$ since $(\underline{T}_0, \underline{F}_0) = (\underline{T}_{s_{\underline{T}_0}}, \underline{F}_{s_{\underline{T}_0}})$. As in Lemma 15(a), this implies that $s = s_{\underline{T}_0}$.

(b) Suppose r is an idempotent radical with $s(G) \leq r(G)$ for all G. Let $(\underline{T}_r, \underline{F}_r)$ be the torsion theory as in Definition 4 and Lemma 11. If $(\underline{T}_{0r}, \underline{F}_{0r})$ is the torsion theory generated by $\underline{T}_r$ and $r_{\underline{T}_{0r}}$ is the $\underline{T}_{0r}$-image preradical then $(\underline{T}_r, \underline{F}_r) = (\underline{T}_{0r}, \underline{F}_{0r})$ by Lemma 10(a) and

$r = r_{\underline{T}_{0r}}$ by part (a) of this lemma. Clearly $\underline{T}_s \subset \underline{T}_r$ since $s(G) \leq r(G)$ for all G. Again using Lemma 10(a), we have $\underline{T}_0 \subset \underline{T}_r$. As in Lemma 15(b) we put all of this together to get, for any G,

$$r(G) = r_{\underline{T}_{0r}}(G) = \overline{\sum_{\substack{h:C\to G \\ C\in \underline{T}_{0r}=\underline{T}_r}} \mathrm{im}(h)} \geq \overline{\sum_{\substack{h:C\to G \\ C\in \underline{T}_0}} \mathrm{im}(h)} = s_{\underline{T}_0}(G)$$

and therefore, $s_{\underline{T}_0} \leq r$. Now, if s is idempotent then $s(G) \in \underline{T}_s \subset \underline{T}_0$ for all G. Thus $s_{\underline{T}_0}(s(G)) \geq \mathrm{im}(s(G) \xrightarrow{1} s(G)) = s(G)$ for all G. By Lemma 3(c) this implies $s \leq s_{\underline{T}_0}$.//

We are now ready to state the dual formulation of our main theorem (omitting part (c) which does not change).

22. Theorem: (a) Every idempotent radical is an image radical.

(b) Every torsion theory has the form $(\underline{T}_s, \underline{F}_s)$ for some image radical s.

Proof: (a) This follows immediately from Lemma 21(a).

(b) This is proved in Lemma 19.//

Radicals and Sufficiency Classes in LCA Groups

2.1 Preliminaries

In this part we deal with specific instances of the ideas defined in part one. We start with the notions of sufficiency class and dual sufficiency class as defined in [A1].

1. Definition: Let H be an LCA group. Then $\underline{S}(H)$ is the class of all LCA groups G with sufficiently many continuous homomorphisms into H to separate the points of G. We call $\underline{S}(H)$ the sufficiency class of H. We call a subgroup of G of the form f(H), where $f:H\to G$, an H-subgroup of G. Then $\underline{S}^*(H)$ is the class of all LCA groups G whose H-subgroups generate a dense subgroup of G. We call $\underline{S}^*(H)$ the dual sufficiency class of H.

Now, we can restate these definitions as follows:

$G \in \underline{S}(H)$ iff $\bigcap_{f:G\to H} \ker(f) = 0$, and $G \in \underline{S}^*(H)$ iff $\overline{\sum_{f:H\to G} \operatorname{im}(f)} = G$.

Restated in this way, it is clear that these classes have been encountered in part one under different names. To make precise their relationship with part one, we define the following preradicals.

2. Definition: Let H be an LCA group. Let $\underline{R_H}$ be the $\{H\}$-kernel radical, $\bigcap_{f:G\to H} \ker(f)$. Let $\underline{R^*_H}$ be the $\{H\}$-image preradical, $\overline{\sum_{f:H\to G} \operatorname{im}(f)}$.

We know immediately from part one that R_H is radical, that R^*_H is idempotent, and that R_H and $R^*_{\hat{H}}$ are dual (see Lemmas 1.13(a), 1.18(b) and 1.18(a) respectively). Thus we are justified in calling R_H the <u>H-radical</u>. In order to indicate the duality present, we call R^*_H the <u>H-coradical</u>. Loosely speaking, $R_H(G)$ is that part of G which cannot be mapped into H. If R_H is not idempotent then this impossibility is dependent on the way in which $R_H(G)$ is embedded in G. Similarly $R^*_H(G)$ is that part of G into which H can be mapped.

Now, there are several facts about $\underline{S}(H)$, $\underline{S}^*(H)$, R_H, and R^*_H which are immediate consequences of the results in part one. We state and prove these below.

<u>Fact</u> <u>A</u>: $\underline{S}(H) = \underline{F}_{R_H}$ and $\underline{S}^*(H) = \underline{T}_{R^*_H}$.

By definition, $G \in \underline{F}_{R_H}$ iff $R_H(G) = 0$. As we remarked before Definition 2, $G \in \underline{S}(H)$ iff $R_H(G) = \bigcap_{f:G \to H} \ker(f) = 0$. Thus, $\underline{S}(H) = \underline{F}_{R_H}$. Similarly, $\underline{S}^*(H) = \underline{T}_{R^*_H}$. This is the relationship between sufficiency classes and the ideas of part one which we mentioned earlier.

For notational simplicity let us denote the class $\underline{T}_{R_H} = \{G: R_H(G) = G\}$ by $\underline{T}(H)$ and the class $\underline{F}_{R^*_H} = \{G: R^*_H(G) = 0\}$ by $\underline{F}^*(H)$.

Fact $\underline{B}$: $\widehat{\underline{S}(H)} = \underline{S}^*(\hat{H})$ and $\widehat{\underline{T}(H)} = \underline{F}^*(\hat{H})$.

In words, the first half says that (G,H) separates points of G iff the $\hat{H}$-subgroups of $\hat{G}$ generate a dense subgroup of $\hat{G}$. To prove this, we note that R_H and $R^*_{\hat{H}}$ are dual, then apply Fact A and Lemma 1.5(d), which states that $\widehat{\underline{F}_r} = \underline{T}_{\hat{r}}$ and $\widehat{\underline{T}_r} = \underline{F}_{\hat{r}}$.

Fact $\underline{C}$: The following are equivalent:

(i) R_H is idempotent

(ii) $R^*_{\hat{H}}$ is radical

(iii) $\underline{S}(H)$ is closed under extensions

(iv) $\underline{S}^*(\hat{H})$ is closed under extensions

(v) $(\underline{T}(H), \underline{S}(H))$ is a torsion theory

(vi) $(\underline{S}^*(\hat{H}), \underline{F}^*(\hat{H}))$ is a torsion theory

By Lemma 1.2(c) and the fact that R_H and $R^*_{\hat{H}}$ are dual, (i) is equivalent to (ii). By Fact A and Lemma 1.6, (i) and (iii) are equivalent and (ii) and (iv) are equivalent. By Fact A and Lemma 1.11, (i) implies (v) and (ii) implies (vi). Finally, by Fact A and Lemmas 1.6 and 1.8, (v) implies (i) and (vi) implies (ii). This proves the assertion.

Note that we may conclude from Lemmas 1.8, 1.13(b), and 1.18(e) that $\underline{T}(H)$ and $\underline{F}^*(H)$ are always closed under extensions.

Fact $\underline{D}$: $R_H(G)$ is the smallest closed subgroup of G which yields a quotient in $\underline{S}(H)$.

Since R_H is radical, $G/R_H(G)$ is in $\underline{S}(H)$ for all LCA groups G. Now, if $G/K \in \underline{S}(H)$ then, by Fact A and Lemma 1.3(a), $R_H(G) \leq K$.

Fact E: $R^*_H(G)$ is the largest closed subgroup of G which is in $\underline{S}^*(H)$.

Since R^*_H is idempotent, $R^*_H(G) \in \underline{S}^*(H)$ for all LCA groups G. If $K \leq G$ and $K \in \underline{S}^*(H)$ then, by Fact A and Lemma 1.3(c), $K \leq R^*_H(G)$.

Fact F: $G \in \underline{T}(H)$ iff $(G,H) = 0$. $G \in \underline{F}^*(H)$ iff $(H,G) = 0$.

Now $G \in \underline{T}(H)$ iff $\bigcap_{f:G \to H} \ker(f) = G$. This is equivalent to the requirement that $\ker(f) = G$ for all $f:G \to H$, which is in turn equivalent to the requirement that $f = 0$ for all $f:G \to H$, that is, $(G,H) = 0$.

Dually, $G \in \underline{F}^*(H)$ iff $\overline{\sum_{f:H \to G} \mathrm{im}(f)} = 0$. This is equivalent to the requirement that $\mathrm{im}(f) = 0$ for all $f:H \to G$, which is equivalent to the requirement that $f = 0$ for all $f:H \to G$, that is, $(H,G) = 0$.

Fact G: $H_1 \in \underline{S}(H_2)$ iff $R_{H_2} \leq R_{H_1}$.

Assume that $H_1 \in \underline{S}(H_2)$. If G is an LCA group and $x \notin R_{H_1}(G)$ then $f(x) \neq 0$ for some $f:G \to H_1$. Since $H_1 \in \underline{S}(H_2)$ and $f(x) \neq 0$, $p(f(x)) \neq 0$ for some $p:H_1 \to H_2$. Then $pf:G \to H_2$ and $pf(x) \neq 0$ so $x \notin R_{H_2}(G)$. Thus $R_{H_2} \leq R_{H_1}$. Now suppose $R_{H_2} \leq R_{H_1}$. Then $R_{H_2}(H_1) \leq R_{H_1}(H_1) = 0$, so $H_1 \in \underline{S}(H_2)$.

<u>Fact</u> <u>H</u>: $H_1 \in \underline{S}^*(H_2)$ iff $R^*_{H_1} \leq R^*_{H_2}$.

The direct proof of one direction is tedious. We prove it by duality. If $H_1 \in \underline{S}^*(H_2)$ then $\widehat{H_1} \in \underline{S}(\widehat{H_2})$ so by Fact G, $R_{\widehat{H_2}} \leq R_{\widehat{H_1}}$. By the note preceding Lemma 1.3 this implies $R^*_{H_1} \leq R^*_{H_2}$. Now, suppose $R^*_{H_1} \leq R^*_{H_2}$. Then $H_1 = R^*_{H_1}(H_1) \leq R^*_{H_2}(H_1)$. Thus $H_1 \in \underline{S}^*(H_2)$.

With these facts in hand, we turn now to a consideration of specific radicals, coradicals, and their associated classes. In some instances the results are well known and we simply restate them in the language of radicals and torsion theories. Also, many of the sufficiency classes and dual sufficiency classes were described in [A1]. For these results we simply make the appropriate reference to [A1].

Most of the radicals and sufficiency classes that are known correspond to divisible groups. This occurs because the divisibility of a group allows us to extend homomorphisms into it, greatly simplifying our task.

2.2 Description of Specific Radicals, Coradicals, and Their Associated Classes

In this section we describe the specific radicals, coradicals and associated classes which we know. We pair the radical and coradical belonging to a particular group, proving all that we know about both of them before moving to the next group. We start with the $\mathbb{R}$-radical and coradical

$\underline{\mathbb{R}}$

3. Proposition: $R_{\mathbb{R}}(G)$ is the subgroup of compact elements of G. $R_{\mathbb{R}}$ is idempotent and $\underline{S}(\mathbb{R})$ is closed under extensions. $\underline{S}(\mathbb{R})$, the class of all LCA groups with no nonzero compact elements, contains an LCA group G iff $G \approx \mathbb{R}^n \times D$, where n is a nonnegative integer and D is a discrete, torsion-free group. $\underline{T}(\mathbb{R})$, the class of all LCA groups with all elements compact, contains exactly those groups whose duals are totally disconnected.

Proof: $R_{\mathbb{R}}(G)$ is the subgroup of compact elements of G by [H&R, 24.34]. Clearly the closed subgroup generated by some $x \in G$ is compact whether x is considered an element of G or of $R_{\mathbb{R}}(G)$. Hence $R_{\mathbb{R}}$ is idempotent and, by Fact C, $\underline{S}(\mathbb{R})$ is closed under extensions. By [H&R, 24.35], $G \in \underline{S}(\mathbb{R})$ iff $G \approx \mathbb{R}^n \times D$ as stated above. The component of the identity in $\hat{G}$ is the annihilator of $R_{\mathbb{R}}(G)$ [H&R, 24.17]. Thus, $R_{\mathbb{R}}(G) = G$ iff $\hat{G}$ is totally disconnected.//

4. Proposition: $R^*_{\mathbb{R}}(G)$ is the component of the identity

in G. $R^*_{\mathbb{R}}$ is radical and $\underline{S}^*(\mathbb{R})$ is closed under extensions. An LCA group G is in $\underline{S}^*(\mathbb{R})$ iff G is connected. Also, G is in $\underline{S}^*(\mathbb{R})$ iff $G \approx \mathbb{R}^n \times E$ where n is a nonnegative integer and E is compact and connected. An LCA group G is in $\underline{F}^*(\mathbb{R})$ iff G is totally disconnected.

Proof: Since $\mathbb{R}$ is self-dual, $R_{\mathbb{R}}$ and $R^*_{\mathbb{R}}$ are dual. Thus, the first two statements follow immediately from [H&R, 24.17], Fact C and Proposition 3. The third statement obviously follows from the first. By [H&R, 9.14], an LCA group is connected iff it has the form $\mathbb{R}^n \times E$ as stated above. Since an LCA group is totally disconnected iff the component of the identity is 0, the last statement is obvious.//

Thus we see that the radicals $R_{\mathbb{R}}$ and $R^*_{\mathbb{R}}$ are familiar canonical subgroups. We now turn to one which is not as familiar but which may be just as important as the two just considered.

$\underline{\hat{Q}}$

5. Proposition: $\underline{S}(\hat{Q})$ is the class of all torsion-free LCA groups. $R_{\hat{Q}}(G)$ is the smallest closed subgroup of G which yields a torsion-free quotient. $R_{\hat{Q}}$ is idempotent and $\underline{S}(\hat{Q})$ is closed under extensions. An LCA group G is in $\underline{T}(\hat{Q})$ iff $\hat{G}$ is reduced.

Proof: The first statement is proved in [A1, Proposition 1]. The second statement follows from the first and Fact D. Now, if H is a closed subgroup of G such that H and G/H are both torsion-free, it is immediate that G is torsion-free. Thus $\underline{S}(\hat{Q})$ is closed under extensions and, by Fact C, $R_{\hat{Q}}$ is idempotent. Now, G is in $\underline{T}(\hat{Q})$ iff G has no nontrivial torsion-free quotients. By [A1, Theorem 1] this is equivalent to the assertion that $\hat{G}$ has no divisible subgroups, which is equivalent to the requirement that $\hat{G}$ be reduced.//

The importance of the $\hat{Q}$-radical lies in the fact that $R_{\hat{Q}}(G)$ is the smallest closed subgroup of G which yields a torsion-free quotient. In discrete groups this characterizes the torsion subgroup (thus $R_{\hat{Q}}(G)$ is the torsion subgroup for discrete groups G). Because of this, $R_{\hat{Q}}(G)$ is one natural generalization of the torsion subgroup.

In [R2, 2.3], the $\hat{Q}$-radical is defined and discussed from a completely different viewpoint. It is obtained by a transfinite iterative process which makes the analogy with the torsion subgroup more apparent. It is obvious from Robertson's definition that $R_{\hat{Q}}(G)=0$ iff G is torsion-free. He calls LCA groups G for which $R_{\hat{Q}}(G)=G$ "transfinite torsion" groups.

Now we look at the $\hat{Q}$-coradical.

6. Proposition: $\underline{S}^*(\hat{Q})$ is the class of all compact connected LCA groups. $R^*_{\hat{Q}}(G)$ is the largest compact connected subgroup of G. $\underline{S}^*(\hat{Q})$ is closed under extensions and $R^*_{\hat{Q}}$ is radical. An LCA group G is in $\underline{F}^*(\hat{Q})$ iff $G \approx \mathbb{R}^n \times G_o$ where n is a nonnegative integer and G_o is totally disconnected.

<u>Proof</u>: The first statement is proved in [A1, Theorem 2]. The second statement follows from the first and Fact E. Now, if H is a closed subgroup of G such that H and G/H are compact and connected then, by [H&R, 7.14 and 5.25], G is compact and connected. Thus $\underline{S}^*(\hat{Q})$ is closed under extensions and, by Fact C, $R^*_{\hat{Q}}$ is radical. Now if G has the form $\mathbb{R}^n \times G_o$ as above then any compact connected subgroup of G must be contained in G_o since $\mathbb{R}^n$ has no nontrivial compact subgroups. But G_o has no nontrivial connected subgroups. Thus $R^*_{\hat{Q}}(G) = 0$ and G is in $\underline{F}^*(\hat{Q})$. Conversely, suppose that G has no nontrivial compact connected subgroups. Then the component of the identity in G is $\mathbb{R}^n$ for some nonnegative integer n by [H&R, 9.14]. But $\mathbb{R}^n$ always splits by [H&R, 24.29]. Thus $G \approx \mathbb{R}^n \times G_o$ for some G_o. Now $G_o \approx G/\mathbb{R}^n$ and $\mathbb{R}^n$ is the identity component of G so by [H&R, 7.3] G_o is totally disconnected.//

<u>Q</u>

We now consider the Q-radical and the Q-coradical. The Q-coradical, the dual of the $\hat{Q}$-radical, will turn out to be important in out consideration of indecomposable LCA groups. It is the analog in LCA groups of the maximal divisible subgroup of a discrete group.

7. <u>Proposition</u>: $\underline{S}(Q)$ is the class of all discrete torsion-free groups. $R_Q(G)$ is the smallest open subgroup of G which yields a torsion-free quotient. $\underline{S}(Q)$ is closed under extensions and R_Q is idempotent. An LCA group G

is in $\underline{T}(Q)$ iff $G \approx \mathbb{R}^n \times G_o$ where n is a nonnegative integer and every element of G_o is compact.

Proof: The first statement is proved in [A1, Proposition 2]. The second statement follows from the first, Fact D, and the fact that a quotient G/H is discrete iff H is open [H&R, 5.26]. Suppose $H \leq G$ such that H and G/H are discrete and torsion-free. Then clearly G is torsion-free. Since G/H is discrete, H is open in G, but since H is discrete this implies that G is discrete. Thus, $\underline{S}(Q)$ is closed under extensions and, by Fact C, R_Q is idempotent. Now G is in $\underline{T}(Q)$ iff $\hat{G}$ is in $\underline{F}^*(\hat{Q})$. But $\hat{G}$ is in $\underline{F}^*(\hat{Q})$ iff $\hat{G} \approx \mathbb{R}^n \times G_o$ with G_o totally disconnected by Proposition 6. Finally, G_o is totally disconnected iff every element of $\hat{G}_o$ is compact. Thus, $G \approx \mathbb{R}^n \times G_o$ with with every element of G_o compact iff G is in $\underline{T}(Q)$.//

Note that if G is discrete then $R_Q(G)$ is the torsion subgroup of G. In this case $R_Q(G) = R_{\hat{Q}}(G)$.

It is remarkable that every group contains a smallest open subgroup with torsion-free quotient. On the face of it, the existence of such a subgroup might seem doubtful since the intersection of a collection of open subgroups is not always open.

8. Proposition: $\underline{S}^*(Q)$ is the class of all densely divisible LCA groups. $R^*_Q(G)$ is the maximal densely divisible subgroup of G. R^*_Q is radical and $\underline{S}^*(Q)$ is closed under extensions. An LCA group G is in $\underline{F}^*(Q)$ iff G is reduced.

Proof: The first statement follows from Proposition 5 and [A1, Theorem 1] by the fact that $\underline{S}^*(Q)$ and $\underline{S}(\hat{Q})$ are dual. The second statement is immediate from Fact E and the first statement. That R^*_Q is radical follows from Fact C and Proposition 5. Then by Fact C, $\underline{S}^*(Q)$ is closed under extensions. It is clear that the maximal densely divisible subgroup of a group is trivial iff that group is reduced.//

By duality and the note preceding Proposition 8, $R^*_Q(G) = R^*_{\hat{Q}}(G)$ in a compact group G.

Note that the maximal densely divisible subgroup is the closure of the maximal divisible subgroup.

The groups $\mathbb{R}$, Q and $\hat{Q}$ are the groups for which we have the most complete results.

Q/Z

In the next theorem we find an interesting phenomenon. Namely, the $\mathbb{R}$-coradical and the Q/Z-radical are equal. This case and its dual are the only cases which we have found in which this occurs. It would be interesting to characterize those pairs of LCA groups, H_1 and H_2, such that the H_1-radical and the H_2-coradical are equal. However, this characterization may have to wait for other examples of this phenomenon. Note that the characterization would have to be self-dual since the $\hat{H}_2$-radical and the $\hat{H}_1$-coradical are equal exactly when the H_1-radical and the H_2-coradical are equal.

9. <u>Proposition</u>: $\underline{S}(Q/Z)$ is the class of all totally disconnected groups. $R_{Q/Z}(G)$ is the identity component of G. Thus $R_{Q/Z} = R^*_{\mathbb{R}}$. $R_{Q/Z}$ is idempotent and $\underline{S}(Q/Z)$ is closed under extensions. $\underline{T}(Q/Z)$ is the class of all connected LCA groups.

<u>Proof</u>: That $\underline{S}(Q/Z)$ is the class of all totally disconnected groups is proved in [A1, Proposition 5]. It follows immediately that $R_{Q/Z}(G)$ is the identity component of G. The other statements then follow immediately from Proposition 4.//

We now consider the Q/Z-coradical.

10. <u>Proposition</u>: $R^*_{Q/Z}(G)$ is the closure of the maximal divisible torsion subgroup of G. $\underline{S}^*(Q/Z)$ is the class of all LCA groups containing a dense subgroup which is a divisible torsion group. $\underline{F}^*(Q/Z)$ is the class of all LCA groups whose torsion subgroup is reduced.

<u>Proof</u>: To prove the first statement we need only show that the maximal divisible torsion subgroup, M, of G equals $\sum_{f:Q/Z\to G} \text{im}(f)$. Clearly $\sum_{f:Q/Z\to G} \text{im}(f) \subset M$ since $\sum_{f:Q/Z\to G} \text{im}(f)$ is a divisible torsion subgroup of G. Now M is algebraically a direct sum of various $Z(p^\infty)$'s [F, 23.1] so there are enough homomorphisms from Q/Z to cover M. Since Q/Z is discrete, these are all continuous no matter what topology M has. Hence $M \subset \sum_{f:Q/Z\to G} \text{im}(f)$, proving equality. Thus $R^*_{Q/Z}(G) = \overline{M}$, proving the first statement. Since $G \in \underline{S}^*(Q/Z)$ iff $R^*_{Q/Z}(G) = G$, the second statement is obvious.

Now $G \in \underline{F}^*(Q/Z)$ iff $R^*_{Q/Z}(G) = 0$ which is equivalent to the requirement that $M = 0$. This occurs exactly when the torsion subgroup of G is reduced.//

Note that it would be equivalent to describe $\underline{F}^*(Q/Z)$ as the class of all LCA groups containing no divisible torsion group.

It would be nice to know whether or [illegible] $_{Q/Z}$ is radical. It seems most likely that $R^*_{Q/Z}$ is radical. We now turn to the group dual to Q/Z.

$\underline{J}$

Note: Recall that J denotes the product $J_2 \times J_3 \times J_5 \times J_7 \times \cdots$ containing one copy of each of the p-adic integer groups, J_p.

11. Proposition: If $G \in \underline{S}(J)$ then G is a torsion-free reduced LCA group.

Proof: If $G \in \underline{S}(J)$ then $\hat{G} \in \underline{S}^*(Q/Z)$ since $\hat{J} \approx Q/Z$. Then by Proposition 10, $\hat{G}$ has a dense divisible torsion subgroup. Since $\hat{G}$ has a dense divisible subgroup, G is torsion-free by [A1, Theorem 1]. We also have that the torsion subgroup, $T(\hat{G})$, is dense, that is, $\overline{T(\hat{G})} = \hat{G}$. By [H&R, 23.24.a] $A(G, T(\hat{G})) = A(G, \overline{T(\hat{G})}) = A(G, \hat{G}) = 0$. Now, by [H&R, 24.24], the maximal divisible subgroup of G is contained in $A(G, T(\hat{G}))$ and is therefore 0. Thus G is reduced.//

At present, the only description we have of $\underline{T}(J)$ is the one obtained by dualizing the description of $\underline{F}^*(Q/Z)$: an LCA group G is in $\underline{T}(J)$ iff the torsion subgroup of $\hat{G}$

is reduced. We also do not know whether or not R_J is idempotent. Our knowledge of the J-coradical is more extensive. In fact, the J-coradical is the $\mathbb{R}$-radical. This is the dual of the equality between the $\mathbb{R}$-coradical and the Q/Z-radical.

12. Proposition: $R^*_J(G)$ is equal to $R_{\mathbb{R}}(G)$, the subgroup of all compact elements of G. R^*_J is radical and $\underline{S}^*(J)$ is closed under extensions. $\underline{S}^*(J)$, the class of all LCA groups with all elements compact, contains an LCA group G iff $\hat{G}$ is totally disconnected. $\underline{F}^*(J)$, the class of all LCA groups with no nonzero compact elements, contains an LCA group G iff $G \approx \mathbb{R}^n \times D$ where n is a non-negative integer and D is a discrete torsion-free group.

Proof: That $R^*_J = R_{\mathbb{R}}$ follows from Proposition 9 and the fact that R^*_J is dual to $R_{Q/Z}$ and $R_{\mathbb{R}}$ is dual to $R^*_{\mathbb{R}}$. The other statements follow from Proposition 3.//

$Z(p^\infty)$

Before we present the following proposition we need a definition. Let p be a prime. An LCA group G is a topological p-group iff $\lim_{n\to\infty} p^n g = 0$ for all $g \in G$ (see [R1, section 3] or [A1, Definition 2]). A p-group is a topological p-group but not all topological p-groups are p-groups. For example, the torsion-free group J_p is a topological p-group but is clearly not a p-group.

13. Proposition: Let p be a prime. An LCA group G is in $\underline{S}(Z(p^\infty))$ iff G is totally disconnected and every compact open subgroup of G is a topological p-group.

Proof: This is Proposition 3 in [A1].//

We know of no nontrivial description of $\underline{T}(Z(p^\infty))$ or $R_{Z(p^\infty)}$.

In the next proposition we use the following notation. If G is an LCA group and p is a prime then $G_p = \{g \in G: p^n g = 0$ for some integer $n\}$ (G_p is called the p-component of G).

14. Proposition: Let p be a prime. $R^*_{Z(p^\infty)}(G)$ is the closure of the maximal divisible p-group contained in G. An LCA group G is in $\underline{S}^*(Z(p^\infty))$ iff G has a dense subgroup which is a divisible p-group. An LCA group G is in $\underline{F}^*(Z(p^\infty))$ iff G_p is reduced. Equivalently, G is in $\underline{F}^*(Z(p^\infty))$ iff G contains no divisible p-group.
Proof: Copy the proof of Proposition 10 substituting p-group for torsion group, p-component G_p for torsion subgroup, and $Z(p^\infty)$ for Q/Z.//

We now consider the dual of $Z(p^\infty)$.

$\underline{J_p}$

The only information we have about the J_p-radical and its associated classes are those statements which are trivially equivalent to the assertion that R_{J_p} is dual to $R^*_{Z(p^\infty)}$. For example, an LCA group G is in $\underline{S}(J_p)$ iff $\hat{G}$ contains a dense divisible p-group. We do not know the duals of the properties mentioned in Proposition 14. Also, part of the difficulty in this case arises because J_p is not divisible. We can, however, dualize the description of $\underline{S}(Z(p^{\infty}))$. That dualization is the substance of the following

15. Proposition: Let p be a prime. An LCA group G is in $\underline{S}^*(J_p)$ iff every element of G is compact and every quotient of G by a compact open subgroup is a (discrete) p-group.

Proof: By Proposition 13 and the duality between $\underline{S}(Z(p^{\infty}))$ and $\underline{S}^*(J_p)$, G is in $\underline{S}^*(J_p)$ iff $\hat{G}$ is totally disconnected and every compact open subgroup of $\hat{G}$ is a topological p-group. By [H&R, 24.17] $\hat{G}$ is totally disconnected iff every element of G is compact. By [R1, 3.18] and [H&R, 23.25 and 23.29.a], every quotient of G by a compact open subgroup is a topological p-group iff every compact open subgroup of $\hat{G}$ is a topological p-group. But quotients by open subgroups are discrete [H&R, 5.26] and discrete topological p-groups are p-groups [R1, 3.2].//

We now take up the self-dual group F_p.

$\underline{F_p}$

Our knowledge of R_{F_p}, $R^*_{F_p}$ and their associated classes is very limited. We can only describe $\underline{S}(F_p)$ and $\underline{S}^*(F_p)$.

16. Proposition: Let p be a prime. An LCA group G is in $\underline{S}(F_p)$ iff G is totally disconnected and every compact open subgroup of G has the form J_p^N for some cardinal number N.

Proof: This is Proposition 4 in [A1].//

Since F_p is self-dual, we obtain a description of

$\underline{S}^*(F_p)$ by dualizing the preceding proposition.

17. Proposition: Let p be a prime. An LCA group G is in $\underline{S}^*(F_p)$ iff every element of G is compact and every quotient of G by a compact open subgroup is topologically isomorphic to a direct sum of $Z(p^\infty)$'s.

Proof: This follows from Proposition 16 exactly as Proposition 15 follows from Proposition 14, using the fact that $(J_p^N)^\wedge \approx \bigoplus_N Z(p^\infty)$, the direct sum of N copies of $Z(p^\infty)$ [H&R, 23.21 and 25.2].//

E(J)

The next group we consider is the minimal divisible extension of $J \approx \widehat{Q/Z}$. Let us denote the minimal divisible extension of an LCA group G by E(G) (see [H&R, 25.32] for a discussion of minimal divisible extensions. Algebraically, E(G) is just the minimal divisible extension of G as a discrete group. The topology given E(G) is the one "inherited" from G which implies that G is open in E(G).) Alternatively, we may describe E(J) as the local direct product of F_p's (one for each prime p) with respect to the compact open subgroups J_p [H&R, 25.32.d]. Let us reserve the symbol "E_o" to stand for E(J) in order to simplify our notation. Slightly more is known about the E_o-radical and coradical than about the preceding few radicals and coradicals.

18. Proposition: $\underline{S}(E_o)$ is the class of all totally disconnected torsion-free LCA groups. $\underline{S}(E_o)$ is closed under extensions and R_{E_o} is idempotent.

Proof: $\underline{S}(E_o)$ is described in [A1, Proposition 6]. By Propositions 5 and 9, $\underline{S}(E_o)$ is closed under extensions. By Fact C, R_{E_o} is idempotent.//

By [H&R, 23.33] E_o is self-dual. Hence, we obtain the major part of the following proposition by duality from the preceding proposition.

19. Proposition: $\underline{S}^*(E_o)$ is the class of all densely divisible LCA groups all of whose elements are compact. $\underline{S}^*(E_o)$ is closed under extensions and $R^*_{E_o}$ is radical. An LCA group G is in $\underline{F}^*(E_o)$ if $R^*_Q(G) \cap R_{\mathbb{R}}(G) = 0$.

Proof: The first statement follows from Proposition 18 by [H&R, 24.17], [A1, Theorem 1] and Fact B since E_o is self-dual. The second statement follows from Fact C and Proposition 18. Now $R^*_{E_o}(G)$ is densely divisible with all elements compact since $R^*_{E_o}$ is idempotent. This implies that $R^*_{E_o}(G)$ is contained in $R^*_Q(G)$ and $R_{\mathbb{R}}(G)$ by Propositions 3 and 8. Thus $R^*_{E_o}(G)$ is contained in their intersection. Thus, if their intersection is 0, $R^*_{E_o}(G) = 0$ and, by Fact F, $G \in \underline{F}^*(E_o)$.//

We consider one more group. It is distinguished by the fact that its coradical is the closure of the torsion subgroup.

$\bigoplus_{n=1}^{\infty} Z(n)$

Let us denote the (discrete) LCA group $\bigoplus_{n=1}^{\infty} Z(n)$ by the symbol "N_o" in order to simplify notation.

20. Proposition: $R^*_{N_o}(G)$ is the closure of the torsion subgroup of G. $\underline{S}^*(N_o)$ is the class of all LCA groups with dense torsion subgroup. $\underline{F}^*(N_o)$ is the class of all torsion-free LCA groups.

Proof: The second statement is mentioned but not proved in [A1] following Proposition 6. Although the proof is quite easy, we include it for completeness. Clearly the first statement will be proved if we show that the torsion subgroup of G is equal to $\sum_{f:N_o \to G} \mathrm{im}(f)$. Obviously, $\sum_{f:N_o \to G} \mathrm{im}(f)$ is contained in the torsion subgroup of G since N_o is torsion. If x is in the torsion subgroup of G and $o(x) = k$ then x is contained in the image of the projection $N_o \to Z(k)$ followed by the topological isomorphism which sends $1 \in Z(k)$ to x. Thus we have proved the first statement. The second and third statements are immediate from the first.//

We now show that $R^*_{N_o}$ is not a satisfactory generalization of the torsion subgroup of a discrete group since it is not even a radical. In the next proposition we use the following notation. If G is an LCA group then nG denotes the subgroup $\{ng: g \in G\}$, that is, the set of elements of G divisible by n.

21. Proposition: $R^*_{N_o}$ is not radical.

Proof: Let G be the group described in [H&R, 24.44.c]. It is shown there that G is reduced and that $\bigcap_{n=1}^{\infty} \overline{nG} \neq 0$.

By [H&R, 24.24, 23.24.a and 23.25] we have

$0 \neq \bigcap_{n=1}^{\infty} \overline{nG} = A(G, R^*_{N_o}(\hat{G})) \approx [\hat{G}/R^*_{N_o}(\hat{G})]^\wedge$ where $R^*_{N_o}(\hat{G})$ is the closure of the torsion subgroup of $\hat{G}$ by Proposition 20. Since G is reduced, $A(G, R^*_{N_o}(\hat{G}))$ is certainly not densely divisible (being nonzero). By [Al, Theorem 1] $\hat{G}/R^*_{N_o}(\hat{G})$ is not torsion-free, so by Proposition 20, $R^*_{N_o}(\hat{G}/R^*_{N_o}(\hat{G})) \neq 0$ and hence $R^*_{N_o}$ is not radical.//

22. Corollary: $R_{\hat{N}_o}$ is not idempotent. $\underline{S}^*(N_o)$ and $\underline{S}(\hat{N}_o)$ are not closed under extensions.

Proof: These follow directly from Proposition 21 by Fact C.//

For convenience of reference, we now list in concise form the canonical subgroups we have discussed with their major properties. We omit only those about which we know nothing. We recall that all H-radicals, R_H, are radical and that all H-coradicals, R^*_H, are idempotent in order to avoid repeating these facts in each case. We group the list into dual pairs. We have included a few results from section 4 for completeness.

We also list the pair of classes associated with each preradical discussed. Again we include a few results from section 4 for completeness. We indicate by an asterisk on the left those which are known to be torsion theories. Again, we group them into dual pairs.

From the number of gaps in the following tables it is apparent that there is still some work to be done on these radicals and their associated classes. We would also like to know whether or not the descriptions given for $\underline{S}(Z(p^{\infty}))$, $\underline{S}^*(J_p)$, $\underline{S}(F_p)$, and $\underline{S}^*(F_p)$ are the simplest possible. They seem rather strong conditions. Perhaps they are equivalent to simpler conditions which appear slightly stronger.

KNOWN CANONICAL SUBGROUPS

$R_{\mathbb{R}}$ = (idempotent) R^*_J (radical)	subgroup containing all compact elements
$R^*_{\mathbb{R}}$ = (radical) $R_{Q/Z}$ (idempotent)	component of the identity
$R_{\hat{Q}}$ (idempotent)	smallest closed subgroup with torsion-free quotient
R^*_Q (radical)	maximal densely divisible subgroup
$R^*_{\hat{Q}}$ (radical)	maximal compact connected subgroup ($R_{\mathbb{R}} \cap R^*_{\mathbb{R}}$)
R_Q (idempotent)	smallest open subgroup with torsion-free quotient ($R_{\mathbb{R}} + R^*_{\mathbb{R}}$)
$R^*_{Q/Z}$	maximal densely divisible torsion subgroup
R_J	??
$R^*_{Z(p^\infty)}$	maximal densely divisible p-subgroup
R_{J_p}	??
$R^*_{N_O}$ (not radical)	closure of the torsion subgroup
$R_{\hat{N}_O}$ (not idempotent)	??
R_{E_O} (idempotent)	smallest closed subgroup with totally disconnected torsion-free quotient ($R_{\hat{Q}} + R^*_{\mathbb{R}}$)
$R^*_{E_O}$ (radical)	maximal densely divisible subgroup containing only compact elements ($R^*_Q \cap R_{\mathbb{R}}$)

KNOWN SUFFICIENCY CLASSES, ETC.

	"TORSION" CLASSES		"TORSION-FREE" CLASSES	
*	$\underline{T}(\mathbb{R}) = \underline{S}^*(J)$	groups with all elements compact	$\underline{S}(\mathbb{R}) = \underline{F}^*(J)$	groups with no compact elements
*	$\underline{S}^*(\mathbb{R}) = \underline{T}(Q/Z)$	connected groups	$\underline{F}^*(\mathbb{R}) = \underline{S}(Q/Z)$	totally disconnected groups
*	$\underline{T}(\hat{Q})$	duals of reduced groups	$\underline{S}(\hat{Q})$	torsion-free groups
*	$\underline{S}^*(Q)$	densely divisible groups	$\underline{F}^*(Q)$	reduced groups
*	$\underline{S}^*(\hat{Q})$	compact connected groups	$\underline{F}^*(\hat{Q})$	groups with non-compact identity component
*	$\underline{T}(Q)$	groups $\mathbb{R}^n \times G_o$ with all elements of G_o compact	$\underline{S}(Q)$	discrete torsion-free groups
*	$\underline{T}(E_o)$	groups G such that $G = R_{\hat{Q}} + R^*_{\mathbb{R}}$	$\underline{S}(E_o)$	totally disconnected torsion-free groups
*	$\underline{S}^*(E_o)$	densely divisible groups with all elements compact	$\underline{F}^*(E_o)$	groups in which $R^*_Q \cap R_{\mathbb{R}} = 0$
	$\underline{S}^*(Q/Z)$	groups containing a dense divisible torsion group	$\underline{F}^*(Q/Z)$	groups whose torsion subgroup is reduced
2	$\underline{T}(J)$	??	$\underline{S}(J)$	contains torsion-free reduced groups
	$\underline{S}^*(Z(p^\infty))$	groups containing a dense divisible p-group	$\underline{F}^*(Z(p^\infty))$	groups with reduced p-component
	$\underline{T}(J_p)$	??	$\underline{S}(J_p)$	??

(continued)

	$\underline{T}(Z(p^{\infty}))$	??	$\underline{S}(Z(p^{\infty}))$	totally disconnected groups in which every compact open subgroup is a topological p-group
	$\underline{S}^*(J_p)$	groups with all elements compact such that every quotient by a compact open subgroup is a topological p-group	$\underline{F}^*(J_p)$	??
	$\underline{T}(F_p)$	??	$\underline{S}(F_p)$	totally disconnected groups in which every compact open subgroup is $\approx J_p^{N}$
	$\underline{S}^*(F_p)$	groups with all elements compact such that every quotient by a compact open subgroup is $\approx \bigoplus_{N} Z(p^{\infty})$	$\underline{F}^*(F_p)$	??
1	$\underline{S}^*(N_o)$	groups with dense torsion subgroup	$\underline{F}^*(N_o)$	torsion-free groups
1	$\underline{T}(\widehat{N_o})$	??	$\underline{S}(\widehat{N_o})$	??

* indicates a torsion theory

1 indicates a pair which is not a torsion theory

2 Note that the description of $\underline{S}(J)$ is not complete.

2.3 Containment Relations

In this section we make heavy use of Facts G and H from section one to investigate the containment relations between the radicals and coradicals introduced in section two. Since most of the proofs involved are nearly trivial, we return to the format of section one, stating our results as "Facts" and proving them with as much dispatch as is possible. We start with two facts which were proved in the previous section.

__Fact__ __CR1__: $R^*_J = R_{\mathbb{R}}$ and $R_{Q/Z} = R^*_{\mathbb{R}}$.

These were proved in Propositions 9 and 12.

We prove a lemma now which will simplify our proofs somewhat.

23. __Lemma__: If $H \leq G$ then $R_G \leq R_H$ and $R^*_{G/H} \leq R^*_G$.
__Proof__: By Lemma 1.5(a,b) and Fact A, $H \in \underline{S}(G)$ and $G/H \in \underline{S}^*(G)$. The result then follows from Facts G and H.//

24. __Corollary__: If $G = H_1 \times H_2$ then $R^*_{H_1} \leq R^*_G$ and $R_G \leq R_{H_1}$.
__Proof__: Since H_1 is a factor of G, it is both a quotient of G and a closed subgroup of G. The result thus follows from Lemma 23.//

__Fact__ __CR2__: $R^*_{Z(p^\infty)} \leq R^*_{Q/Z}$, $R_{Q/Z} \leq R_{Z(p^\infty)}$, $R^*_{F_p} \leq R^*_{E_o}$, $R_{E_o} \leq R_{F_p}$, $R^*_{J_p} \leq R^*_J$, and $R_J \leq R_{J_p}$ for any prime p.

It is obvious that $Z(p^\infty)$ is a factor of Q/Z, that F_p is a factor of E_o, and that J_p is a factor of J. Thus these all follow immediately from Corollary 24.

<u>Fact</u> <u>CR3</u>: $R_{E_o} \leq R_J$, $R^*_{Q/Z} \leq R^*_{E_o}$, $R_{F_p} \leq R_{J_p}$, and $R^*_{Z(p^\infty)} \leq R^*_{F_p}$ for any prime p.

These follow from Lemma 23 and the fact that $E_o/J \approx Q/Z$ and $F_p/J_p \approx Z(p^\infty)$.

<u>Fact</u> <u>CR4</u>: $R^*_{\hat{Q}} \leq R^*_{F_p} \leq R^*_{J_p}$ and $R_{Z(p^\infty)} \leq R_{F_p} \leq R_Q$ for any prime p.

Now, $\hat{Q}$ is compact and has no proper open subgroups so by Proposition 17, $\hat{Q} \in \underline{S}^*(F_p)$. Now F_p is compact and every quotient of F_p by a compact open subgroup is topologically isomorphic to $Z(p^\infty)$ so by Proposition 15, $F_p \in \underline{S}^*(J_p)$. Thus the first result follows by Fact H. The other result follows by duality from the remark following Lemma 1.2 (that if $r \leq s$ then $\hat{s} \leq \hat{r}$).

<u>Fact</u> <u>CR5</u>: $R^*_{E_o} \leq R^*_Q$ and $R_Q \leq R_{E_o}$.

Since E_o is divisible, $E_o \in \underline{S}^*(Q)$ by Proposition 8. The results follow by Fact H and duality.

<u>Fact</u> <u>CR6</u>: $R^*_{N_o} \leq R_{\hat{Q}}$ and $R^*_Q \leq R_{\hat{N_o}}$.

Now $R^*_{N_o}$ is the closure of the torsion subgroup by Proposition 20, and $G/R_{\hat{Q}}(G)$ is torsion-free by Proposition 5. Thus $R_{\hat{Q}}$ contains the torsion subgroup, and since $R_{\hat{Q}}$ is closed, it contains $R^*_{N_o}$. The other result follows by duality.

Fact CR7: $R_{Q/Z} \leq R_{E_o}$ and $R^*_{E_o} \leq R^*_J$.

By Proposition 18, E_o is totally disconnected. By Proposition 9, $E_o \in \underline{S}(Q/Z)$. The result follows by Fact G and duality.

Fact CR8: $R_{\hat{Q}} \leq R_{\mathbb{R}} \leq R_Q$ and $R^*_{\hat{Q}} \leq R^*_{\mathbb{R}} \leq R^*_Q$.

Now $\mathbb{R}$ is torsion-free so by Propositi[illegible] $\mathbb{R} \in \underline{S}(\hat{Q})$. Clearly Q has no nonzero compact elements. Thus by Proposition 3, $Q \in \underline{S}(\mathbb{R})$. Then by Fact G and duality the result follows.

Fact CR9: $R^*_{Q/Z} \leq R^*_{N_o}$ and $R_{\hat{N}_o} \leq R_J$.

Clearly any image of Q/Z is torsion and is therefore contained in $R^*_{N_o}$, the closure of the torsion subgroup. Since $R^*_{N_o}$ is closed, $R^*_{Q/Z} \leq R^*_{N_o}$. The second result follows by duality.

Now, the obvious containments can be derived from Facts CR1 through CR9. We have taken care to prove a minimal subset of them in the following sense. Suppose it is true that $A \leq B \leq C$. Then we proved that $A \leq B$ and $B \leq C$ but we did not prove that $A \leq C$ since this can be obtained by transitivity. We will construct a graph later in this section which will display the implied containments clearly. However, a few comments are in order first. We have no guarantee that the containment results proved so far are complete. There are two other types of results which it would be necessary to prove for each of the preradicals we are considering in order to complete these results. First, for each preradical r, we should prove that the only preradicals greater than or equal to r (among those we are considering) are those we have proved to be greater than or equal to r. Facts G and H of section one would be useful in this since it is often elementary to determine whether or not a given LCA group is contained in a specific sufficiency class or dual sufficiency class. As an example, we prove

<u>Fact</u> <u>CR10</u>: Let p be a prime. None of the following are true: (i) $R^*_{Q/Z} \leq R^*_{F_p}$, (ii) $R^*_{F_p} \leq R^*_{Q/Z}$, (iii) $R_J \leq R_{F_p}$, (iv) $R_{F_p} \leq R_J$.

We see that (iii) and (iv) are dual to (i) and (ii) so that we need only prove (i) and (ii) false. It is clear from Propositions 17 and 10 that $Q/Z \notin \underline{S}^*(F_p)$ since every quotient of Q/Z by a compact open subgroup is topologically isomorphic to Q/Z, and that $F_p \notin \underline{S}^*(Q/Z)$ since F_p is torsion free. Fact H then completes the proof.

If the class needed to carry out this sort of proof has not been characterized (e.g., $\underline{S}(J_p)$) a counterexample would be necessary.

The second type of result which should be proved is that each containment is in general strict. Clearly only pairs, $r \leq s$, for which there is no preradical between r and s (among the preradicals being considered), need be checked. Facts G and H are as useful in this task as in the first one. Counterexamples are sometimes needed. For example, as a simple corollary to Proposition 21 we have the following

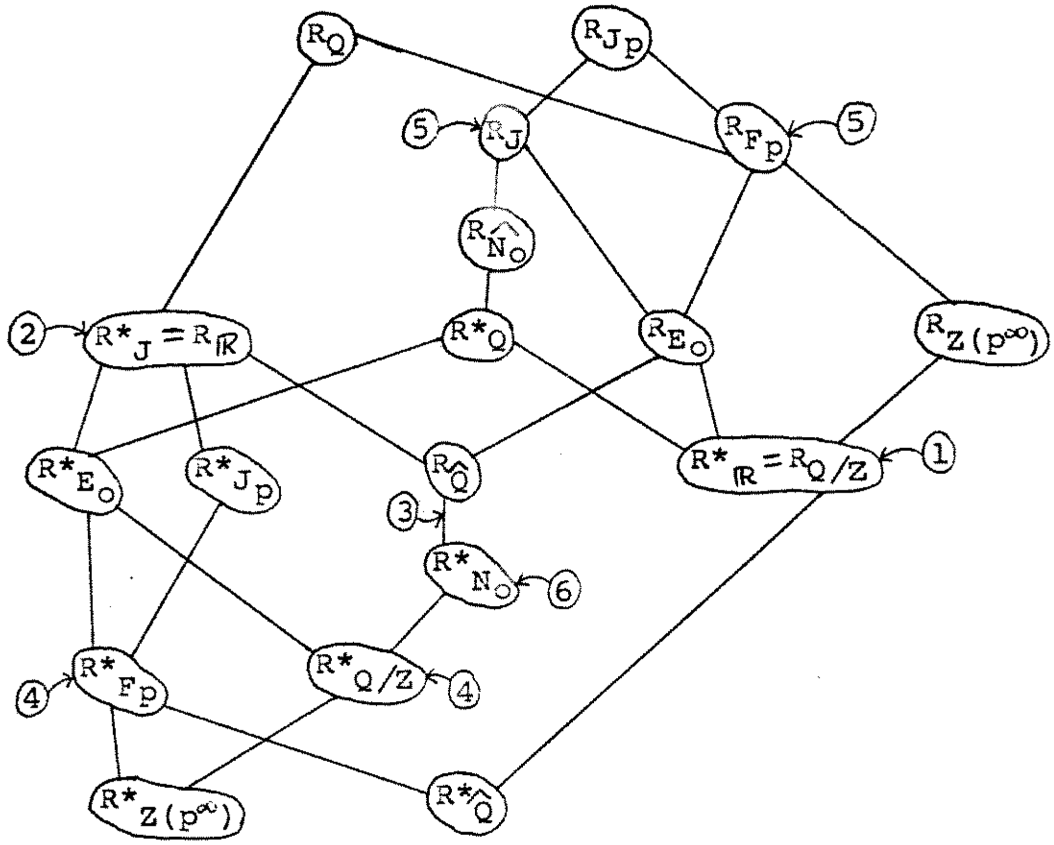
<u>Fact</u> <u>CR11</u>: $R^*_{N_0} \neq R_{\hat{Q}}$.

By Proposition 21, $R^*_{N_0}$ is not radical. Since $R_{\hat{Q}}$ is radical, the result is obvious.

Since it is unlikely that any significant results would be obtained by completing the containment results in the manner indicated, we avoid that onerous task and content ourselves with the knowledge that it can be done in a relatively effecient manner should it ever seem desirable.

We collect Facts CR1 through CR11 in graphical form on the following page for easy reference. The arrangement is such that the preradical r is contained in the preradical s iff s can be reached from r by lines which lead toward the top of the graph.

Known Containment Relations Between Preradicals in an Arbitrary LCA Group



$J = J_2 \times J_3 \times J_5 \times \cdots = \prod_p J_p$ $\qquad E_o = E(J)$ $\qquad N_o = \bigoplus_{n=1}^{\infty} Z(n)$

(1) Component of the identity

(2) Subgroup of compact elements

(3) Strict containment

(4) Not comparable

(5) Not comparable

(6) Closure of the torsion subgroup

Figure 1

2.4 Miscellaneous Results

In this section we prove various further results about the radicals, coradicals and associated classes studied in sections two and three.

Our first result lends considerable weight to the feeling that $R_{\hat{Q}}$ is the natural generalization for LCA groups of the torsion subgroup of a discrete group. Recall from Proposition 20 and Facts CR6 and CR11 that $R_{\hat{Q}}$ strictly contains the closure of the torsion subgroup. We now have the following

25. <u>Theorem</u>: $R_{\hat{Q}}$ is the smallest radical which contains the torsion subgroup.

<u>Proof</u>: If r is a radical then $r(G/r(G)) = 0$ for any LCA group G. If r contains the torsion subgroup then $G/r(G)$ must be torsion-free. By Proposition 5, $G/r(G) \in \underline{S}(\hat{Q})$. Thus, by Fact D, $R_{\hat{Q}} \leq r$. Since $R_{\hat{Q}}$ is radical, it is the smallest radical containing the torsion subgroup.//

The requirement that r be radical in Theorem 25 is not actually a restriction since we certainly want a "generalized torsion subgroup" to yield a quotient whose "generalized torsion subgroup" is 0, which is just another way of saying that we want it to be radical. We will return to the $\hat{Q}$-radical in part three.

We now prove two results (and their duals) concerning the relationship between various radicals and coradicals.

26. Proposition: $R^*_{\hat{Q}}(G) = R_{\mathbb{R}}(G) \cap R^*_{\mathbb{R}}(G)$ and $R_Q(G) = R_{\mathbb{R}}(G) + R^*_{\mathbb{R}}(G)$ for any LCA group G.

Proof: By Proposition 4, $R^*_{\mathbb{R}}(G)$ is the identity component of G. By [H&R, 9.14] it must have the form $\mathbb{R}^n \times E$ where E is compact and connected. Now $R_{\mathbb{R}}(G)$ is the subgroup of compact elements of G by Proposition 3. No nonzero element of $\mathbb{R}$ is compact and, since E is compact, every element of E is compact. Thus $R_{\mathbb{R}}(G) \cap R^*_{\mathbb{R}}(G) = E$. It is clear that E is the maximal compact connected subgroup of G, so the first equality follows by Proposition 6. Dualizing (taking annihilators), the second equality follows by [H&R, 23.29.b] since $R_{\mathbb{R}}(G) + R^*_{\mathbb{R}}(G)$ is always open and thus closed by [H&R, 9.26.a and 5.5].//

We can derive a couple of interesting corollaries from this result.

27. Corollary: If r is any preradical such that $R^*_{\mathbb{R}} \leq r \leq R_Q$ then $R_Q(G) = R_{\mathbb{R}}(G) + r(G)$ and $R^*_{\hat{Q}}(G) = R^*_{\mathbb{R}}(G) \cap \hat{r}(G)$ for all LCA groups G. In particular, this holds if r is R_{E_o}, $R_{Z(p^\infty)}$, or R_{F_p} (then $\hat{r}$ is $R^*_{E_o}$, $R^*_{J_p}$, or $R^*_{F_p}$ respectively).

Proof: Since $r \leq R_Q$, clearly $R_Q(G) = R_Q(G) + r(G)$ for any LCA group G. By the preceding proposition then, $R_Q(G) = R_{\mathbb{R}}(G) + R^*_{\mathbb{R}}(G) + r(G)$. Since $R^*_{\mathbb{R}} \leq r$, $R_Q(G) = R_{\mathbb{R}}(G) + r(G)$. Since $R_Q(G)$ is closed, the second equality follows as in Proposition 26. Clearly the preradicals named satisfy the requirement (see Figure 1).//

Obviously a similar result holds for preradicals between $R_{\mathbb{R}}$ and R_Q. It was not stated because we do not have any examples of such preradicals. The next result is a trivial consequence of Corollary 27. However, it is sufficiently interesting and is obtained with so little work that it deserves mention.

28. Corollary: Let r be as in Corollary 27. Then $R_{\mathbb{R}}(G) + r(G)$ is open and $R^*_{\mathbb{R}}(G) \cap \hat{r}(G)$ is compact in every LCA group G.

Proof: $R_Q(G)$ is open by Proposition 7. The result then follows from Corollary 27 and the fact that the annihilator of an open subgroup is compact [H&R, 23.29.a].//

We now have a result similar to Proposition 26.

29. Proposition: $R_{E_o}(G) = R_{\hat{Q}}(G) + R^*_{\mathbb{R}}(G)$ and $R^*_{E_o}(G) = R^*_Q(G) \cap R_{\mathbb{R}}(G)$ for any LCA group G.

Proof: By [R2, 4.6 and 4.7] G can be written in the form $\mathbb{R}^n \times G_o$ where G_o has no subgroups topologically isomorphic to $\mathbb{R}$. We will consider G_o first. Let us write $A = R_{\hat{Q}}(G_o)$ and $C = R^*_{\mathbb{R}}(G_o)$ for notational simplicity. In the previous section it was shown that $R_{E_o}(G_o)$ contains A and C and therefore their sum. Now C is compact by [H&R, 9.14] since G_o has no subgroups topologically isomorphic to $\mathbb{R}$. Thus, by [H&R, 4.4], $A + C$ is closed. Recalling that C is the component of the identity (Proposition 4), we see that $G_o/(A + C)$ is totally disconnected by [H&R, 7.3 and 7.11]. Now we will show that $G_o/(A + C)$

is torsion-free. Suppose that $mg \in A + C$, say $mg = a + c$ with $a \in A$ and $c \in C$ for some $g \in G_o$. Then, since C is divisible [H&R, 24.24], we have $m(g - c_o) = a$ for some $c_o \in C$. Since G_o/A is torsion-free (Proposition 5) this implies that $g = a_o + c_o$ for some $a_o \in A$. Thus $g \in A + C$ and $G_o/(A + C)$ is therefore torsion-free. Since $G_o/(A + C)$ is torsion-free and totally disconnected, $R_{E_o}(G_o) \subseteq A + C$ by Fact D and Proposition 18. Thus, $R_{E_o}(G_o) = R_{\hat{\mathbb{Q}}}(G_o) + R^*_{\mathbb{R}}(G_o)$. Now, $R_{E_o}(G) = \mathbb{R}^n + R_{E_o}(G_o)$ by Proposition 18 since $\mathbb{R}$ is connected. Also, $R_{\hat{\mathbb{Q}}}(G) = R_{\hat{\mathbb{Q}}}(G_o)$ by Proposition 5 since $\mathbb{R}$ is torsion-free. Finally, $R^*_{\mathbb{R}}(G) = \mathbb{R}^n + R^*_{\mathbb{R}}(G_o)$ by Proposition 4 since $\mathbb{R}$ is connected (or simply recall the definition of $R^*_{\mathbb{R}}$). Thus, we have $R_{E_o}(G) = R_{E_o}(G_o) + \mathbb{R}^n = R_{\hat{\mathbb{Q}}}(G_o) + R^*_{\mathbb{R}}(G_o) + \mathbb{R}^n = R_{\hat{\mathbb{Q}}}(G) + R^*_{\mathbb{R}}(G)$. Dualizing, we obtain the second equality by [H&R, 24.42].//

Notice that this implies that $R_{\hat{\mathbb{Q}}}(G) + R^*_{\mathbb{R}}(G)$ is closed in any LCA group G. Obviously a result analogous to Corollary 27 could be proved but since we know of no preradicals between R_{E_o} and $R_{\hat{\mathbb{Q}}}$ or between R_{E_o} and $R^*_{\mathbb{R}}$ it is of little interest.

We can use Proposition 29 to obtain a complete characterization of groups with trivial E_o-coradical. This characterization is the substance of

30. <u>Proposition</u>: The following are equivalent for any LCA group G: (a) $R^*_{E_o}(G) = 0$

(b) the maximal densely divisible subgroup $R^*_{\mathbb{Q}}(G)$, contains no compact elements, Note: $R^*_{E_0}(G) = 0$ iff G contains no homomorphic images of p-adic numbers Ω_p for all p.

Thus: Thm 31 says:

Either (1) $\exists f: \Omega_p \to G$ with $F \neq 0$

or (2) $G = \mathbb{R}^n \times (\bigoplus_{\mathbb{N}} Q) \times G_0$ with G_0 tot. disc. and red.

[ie, $\mathcal{R}^*_{\mathbb{R}}(G) = \mathbb{R}^n$ comp. of the ident. and $\mathcal{R}^*_{Q}(G) = \mathbb{R}^n \times (\bigoplus_{\mathbb{N}} Q)$ max divis. subgp

both split]

that is, $R^*_Q(G) \cap R_{\mathbb{R}}(G) = 0$

(c) $R^*_Q(G) \approx \mathbb{R}^n \times D$ where n is a nonnegative integer and D is a discrete divisible torsion-free group.

<u>Note</u>: At first it might seem that this could have been proved in Proposition 19 without the use of Proposition 29. The difficulty, however, lies in the fact that we had no guarantee that $R^*_Q(G) \cap R_{\mathbb{R}}(G)$ would be densely divisible until we proved Proposition 29.

<u>Proof</u>: The equivalence of (a) and (b) follows directly from Proposition 29. The equivalence of (b) and (c) follows immediately from Proposition 3.//

We can obtain from Proposition 30 a splitting theorem.

31. <u>Theorem</u>: If G is an LCA group such that $R^*_{E_0}(G) = 0$ then $G \approx \mathbb{R}^n \times D \times G_0$ where n is a nonnegative integer, $D = \bigoplus_{N} Q$ for some cardinal number N, and G_0 is totally disconnected and reduced. That is, the component of the identity and the maximal densely divisible subgroup both split.

<u>Proof</u>: By the preceding proposition, $R^*_Q(G) \approx \mathbb{R}^n \times D$ where D is a direct sum of copies of Q by [F, 23.1]. Since the component of the identity is contained in R^*_Q (Fact CR8 and Proposition 4) we see that it is $\mathbb{R}^n$. Since $\mathbb{R}^n$ always splits [R2, 4.6] we have $G \approx \mathbb{R}^n \times G_1$ with G_1 totally disconnected. Clearly $D \le G_1$. Since G_1 is totally disconnected, D can be split from it by [R2, 4.23]. Thus,

$G \approx \mathbb{R}^n \times D \times G_o$. Since $\mathbb{R}^n$ is the component of the identity, G_o is totally disconnected [H&R, 7.3], and since $\mathbb{R}^n \times D$ is the maximal densely divisible subgroup, G_o is reduced (Proposition 8).//

We know of no other results analogous to Propositions 26 and 29. However, it would not be surprising to find that some more exist.

We now consider the containment relations between our radicals and coradicals in an important special case. We will use a structure theorem due to Robertson [R2, 5.2]. Using our notation, his theorem is the following.

32. <u>Theorem</u>(Robertson): Let G be an LCA group. Then $G \approx \mathbb{R}^n \times (\bigoplus_N Q) \times \hat{Q}^M \times H$ where n is a nonnegative integer, N and M are arbitrary cardinals, and H does not have any subgroups topologically isomorphic to Q or $\mathbb{R}$, or quotient groups topologically isomorphic to $\hat{Q}$. Moreover, $R_{\mathbb{R}}(H)$ is open in H, $R^*_{\mathbb{R}}(H) \leq R_{\hat{Q}}(H)$ and $R^*_Q(H) \leq R_{\mathbb{R}}(H)$.
<u>Note</u>: By [R2, 2.6] Robertson's G_D is $R^*_Q(G)$ and by [R2, 2.4] his G_T is $R_{\hat{Q}}(G)$.

By restricting our attention to H as above we can obtain much stronger containment results than we can in a general LCA group. By use of this theorem we can often reduce our considerations to the group H. At this point the stronger containment relations are very useful. We can derive three equalities in addition to the containments mentioned in Theorem 32.

33. <u>Lemma</u>: Let H be as in Theorem 32. Then:

(a) $R_{\mathbb{R}}(H) = R_{\mathbb{Q}}(H)$

(b) $R^*_{\hat{\mathbb{Q}}}(H) = R^*_{\mathbb{R}}(H)$

(c) $R_{\hat{\mathbb{Q}}}(H) = R_{E_0}(H)$

(d) $R^*_{\mathbb{Q}}(H) = R^*_{E_0}(H)$

<u>Proof</u>: (a) By Theorem 32, $R_{\mathbb{R}}(H)$ is open. Hence $H/R_{\mathbb{R}}(H)$ is discrete [H&R, 5.26]. By Proposition 3, $H/R_{\mathbb{R}}(H)$ is torsion-free. Hence $R_{\mathbb{Q}}(H) \leq R_{\mathbb{R}}(H)$ by Fact D. Since $R_{\mathbb{R}} \leq R_{\mathbb{Q}}$ (Fact CR8) this forces equality.

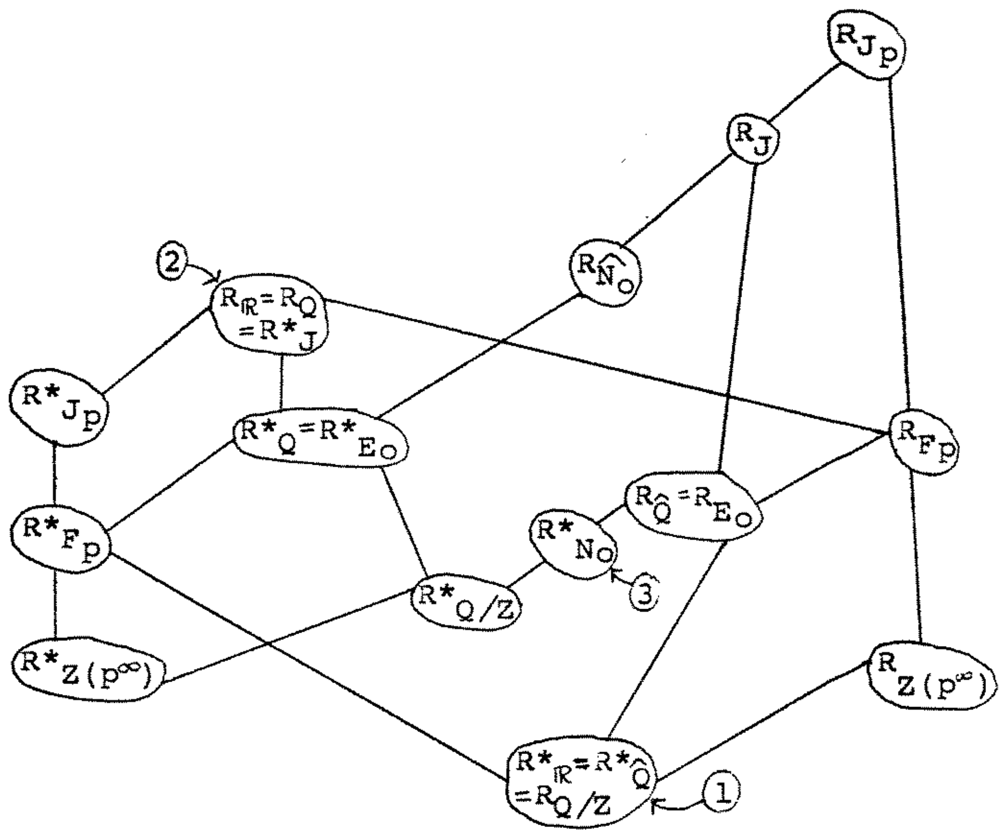
(b) If H is as in Theorem 32 then the identity component $R^*_{\mathbb{R}}(H)$ is compact by [H&R, 9.14] since H has no subgroups topologically isomorphic to $\mathbb{R}$. Thus $\hat{H}$ satisfies the conditions imposed on H in Theorem 32. Thus (b) follows by duality from (a) applied to $\hat{H}$.

(c) Since the identity component $R^*_{\mathbb{R}}(H)$ is contained in $R_{\hat{\mathbb{Q}}}(H)$, the quotient $H/R_{\hat{\mathbb{Q}}}(H)$ is totally disconnected by [H&R, 7.3 and 7.11]. By Proposition 5, $H/R_{\hat{\mathbb{Q}}}(H)$ is torsion-free. Hence, by Proposition 18 and Fact D, $R_{E_0}(H) \leq R_{\hat{\mathbb{Q}}}(H)$ which implies their equality by Fact CR5.

(d) As in (b) this follows by duality from (c).//

These relations are presented in compact form on the following page exactly as in Figure 1.

Known Containment Relations in an LCA Group H as in Theorem 32 (H has no subgroups $\approx Q$ or $\mathbb{R}$, no quotients $\approx \hat{Q}$)



$J = J_2 \times J_3 \times J_5 \times \cdots \approx \prod_p J_p$ $\qquad E_o = E(J)$ $\qquad N_o = \bigoplus_{n=1}^{\infty} Z(n)$

① Component of the identity - compact

② Subgroup of compact elements - open

③ Closure of the torsion subgroup

Figure 2

The following fact is interesting and might be useful at some time. If $(\underline{T},\underline{F})$ is a torsion theory and r is the corresponding radical then any LCA group G is an extension of a unique element of $\underline{T}$ (namely, r(G)) by a unique element of $\underline{F}$ (namely, G/r(G)). Put in other terms, if $H \leq G$ such that $H \in \underline{T}$ and $G/H \in \underline{F}$ then $H = r(G)$. This is an immediate consequence of Lemma 1.3(a,c). This fact is simply a generalization of several well known facts. For example, the identity component is the unique connected subgroup which yields a totally disconnected quotient. The reason this fact has not been very useful yet may be that the torsion theories about which we know a fair amount are associated with exactly those canonical subgroups which are well known and whose properties have already been exploited to a considerable degree.

The Structure of LCA Groups - Open Problems

In this part we consider various questions about the structure of LCA groups and prove as much as we now know. None of the answers are really complete. We present a number of questions which are interesting in themselves or which seem necessary to answer before further progress can be made on the intrinsically interesting questions. We consider in the first two sections embedding and compactifications, and indecomposability and splitting. The last section contains primarily questions of varied nature.

Whatever else its merit, the third part of this paper certainly proves that research in LCA groups will not come to a halt because of a lack of interesting unsolved problems anytime in the forseeable future.

3.1 Embedding and Compactification

Let H be a compact LCA group. Given an arbitrary LCA group G, we will embed a continuous homomorphic image of G in a product of H's. In fact, it will be the largest such image for which this is possible.

We define the function $p:G\to\prod H_{(f)}$, where f ranges over (G,H), each $H_{(f)}\approx H$, by $p(g)=(f(g))_{f\in(G,H)}$ for all $g\in G$. That is, if f_i is a particular map $G\to H$ then the f_i-th coordinate of p(g) is just $f_i(g)$. Now, it is obvious that p is a homomorphism since each $f\in(G,H)$ is a homomorphism and the group operation in $\prod H_{(f)}$ is coordinatewise. Now, p is also continuous; suppose $U=U_{(f_1)}\times U_{(f_2)}\times\cdots\times U_{(f_k)}\times\prod H_{(f)}$ (where f ranges over the elements of (G,H) not in $\{f_1,\ldots,f_k\}$ in the last product) is a nhood of 0 in $\prod H_{(f)}$. By the continuity of f_i, there exists a nhood V_i of 0 in G such that $f_i(V_i)\subset U_{(f_i)}$ for each $i=1,\ldots,k$. Clearly $\bigcap_{i=1}^{k} V_i$ is a nhood of 0 in G and $p(\bigcap_{i=1}^{k} V_i)\subset U$. Thus p is a continuous homomorphism.

We now have the following

1. Proposition: p(G) is a continuous isomorphic image of $G/R_H(G)$. $p(G)\approx G/R_H(G)$ iff $G/R_H(G)$ is compact.

Proof: It is immediate that $\ker(p)=R_H(G)$ since $g\in\ker(p)$ iff $p(g)=0$ which is equivalent to the assertion that $f(g)=0$ for all $f\in(G,H)$ which is true iff $g\in\bigcap_{f:G\to H}\ker(f)=$

$R_H(G)$. Thus, if we define $p^*:G/R_H(G)\to p(G)$ by $p^*(g+R_H(G)) = p(g)$ then p^* is an algebraic isomorphism. Letting $u:G\to G/R_H(G)$ be the natural map we have $p^*u = p$. Since p is continuous and u is open, p^* is continuous, proving the first assertion. Now, if $p(G)\approx G/R_H(G)$ then $p(G)$ is locally compact since $G/R_H(G)$ is. Thus $p(G)$ is closed in $\prod H_{(f)}$ by [H&R, 5.11] and, since $\prod H_{(f)}$ is compact, $p(G)$ is compact. Then $G/R_H(G)$ is compact since it is topologically isomorphic to $p(G)$. Now suppose $G/R_H(G)$ is compact. Then p^* is a 1-1 continuous map from a compact set to a Hausdorff set and is therefore a homeomorphism.//

Clearly $G/R_H(G)$ is the largest image of G (i.e., the image with the smallest kernel) which can be mapped by a continuous homomorphism into any product $\prod H_i$ with each $H_i \approx H$. (If $s:G\to\prod H_i$ then $r_i s \in (G,H)$, where r_i is the i-th projection map. Thus $R_H(G)\leq \ker(r_i s)$ for each r_i and therefore $R_H(G)\leq \bigcap\ker(r_i s) = \ker(s)$.)

Now if we let $G_H = \overline{p(G)}$ then G_H is a compactification of $G/R_H(G)$. Note that $G_H = (G/R_H(G))_H$. This is a generalization of the Bohr compactification. If H is taken to be T, the circle group, then G_T is the Bohr compactification bG [H&R, 26.11]. Actually, it turns out that this does not really generalize the Bohr compactification at all. In fact, we have the

2. Proposition: The compactification G_H of G is topologically isomorphic to a Bohr compactification of G and of $G/R_H(G)$.

Proof: In [H&R, 26.13] it is proved that if H is compact and if p is a continuous homomorphism from G onto a dense subgroup of H, then H is a Bohr compactification of G. Since $G_H = (G/R_H(G))_H$, G_H is a Bohr compactification of both G and $G/R_H(G)$.//

This does not mean that we might as well forget the compactifications G_H. It is difficult to determine exactly which Bohr compactification is topologically isomorphic to G_H. Also, we may be assured that G_H has certain desired properties by its construction. For example, the $\hat{Q}$ compactification is always torsion-free. In general, $G_H \in \underline{S}(H)$ for any G whatsoever.

Now, if H is not compact, the above procedure cannot be carried out since only finite products of H will be locally compact. However, if H has compact open subgroups, it may be possible to carry out an analogous construction using local direct products rather than full direct products. If that is possible then an embedding is possible. However, it is unlikely that a compactification will result since a local direct product is compact only if it is a full direct product.

Problem 1: Carry out such a construction for non-compact H or show that it is impossible. Investigate the properties of the embedding that results if it is possible.

Note that if $H = \mathbb{R}$, for example, this construction is certainly not possible since $\mathbb{R}$ has no compact open subgroups.

Problem 2: Investigate the properties of the compactifications G_H for compact H. Compute some examples if possible.

Conjecture: The compactification G_H of G is topologically isomorphic to the Bohr compactification of G obtained from the set of characters in $\hat{G}$ which factor through H (that is, those characters $\psi \in \hat{G}$ such that $\psi = \gamma\alpha$ for some $\alpha \in (G,H)$ and some $\gamma \in \hat{H}$).

3.2 Indecomposability and Splitting

In this section we prove results asserting that under certain conditions LCA groups split into a direct product or direct sum. We also state and use several important results which are proved elsewhere concerning the splitting of LCA groups. The splitting results are of interest in themselves and they help elucidate the structure of LCA groups. However, they are also useful in characterizing the indecomposable LCA groups.

There are two particular splitting theorems from the theory of discrete (abelian) groups which we would like to generalize to LCA groups. They are:

(i) a discrete abelian group is the direct sum of a divisible group and a reduced group [F,21.3].

(ii) (Kulikov) An indecomposable discrete abelian group is either torsion-free, $Z(n)$ or $Z(p^\infty)$ [F,27.4].

These results imply that an indecomposable discrete abelian group is either $Z(p^n)$, $Z(p^\infty)$, torsion-free divisible, or torsion-free reduced.

Before we can state plausible generalizations of these theorems we must decide how to generalize the maximal divisible subgroup and the torsion subgroup. From the results of part two, especially Theorem 2.25 and the remark following Theorem 2.8, it is fairly clear that R^*_Q is the natural analog of the maximal divisible subgroup and that $R_{\hat{Q}}$ is the natural analog of the torsion subgroup. Now, an example of one of the most pleasing aspects of the theory of LCA groups appears. These two subgroups are dual to one another! Thus $R^*_Q(G)$ splits from G iff $R_{\hat{Q}}(\hat{G})$ splits from $\hat{G}$. Recalling that $R_{\hat{Q}}(G)$ is the torsion subgroup of G if G is discrete, we see immediately that the analog for LCA groups of (i) is not true in general since there are discrete groups whose torsion subgroup does not split. The truth or falsity of the analog of (ii) is not as easily settled. In fact, we do not know whether or not it is true. By the end of this section we will have proved as much as we now know about it. We now state it as a conjecture for easy reference.

3. Conjecture: Let G be an LCA group. If $R_{\hat{Q}}(G)$ is a proper subgroup of G then G is decomposable. Dually, if $R^*_Q(G)$ is a proper subgroup of G then G is decomposable.

The first statement says that an indecomposable LCA group either has no torsion-free quotients or is torsion-free itself. The dual formulation is equivalent to the statement that an indecomposable LCA group is either densely divisible or reduced. We will find this a convenient form of the conjecture to deal with, primarily because of a theorem of Robertson's [R2, 6.4]. There is one other plausible analog of (ii) which has not been disproved. For completeness, we state it as a

4. Conjecture: Let G be an LCA group. If $R_{\mathbb{R}}(G)$ is a proper subgroup of G, then G is decomposable. Dually,

if $R^*_{\mathbb{R}}(G)$ is a proper subgroup of G then G is decomposable.

The first statement of the conjecture says that an indecomposable LCA group either has all elements compact or it has no compact elements, in which case it must either be $\mathbb{R}$ or a discrete torsion-free group (Proposition 2.3). The dual statement says that an indecomposable LCA group is either connected (in which case it must be $\mathbb{R}$ or a compact connected group) or totally disconnected (Proposition 2.4).

It is known that none of the subgroups $R_{\hat{Q}}(G)$, $R^*_Q(G)$, $R_{\mathbb{R}}(G)$ and $R^*_{\mathbb{R}}(G)$ always split [R2, 4.10]. This casts no doubt on the conjectures however, since the torsion subgroup does not always split from a discrete group, yet (ii) is true.

We begin with some results about splitting in compact groups obtained primarily by duality from the discrete case. Next we present several results concerning splitting in general LCA groups. Finally, we will finish this section by proving as much as we can about Conjecture 3.

We begin our investigation of compact groups with the result which is dual to (i).

5. Proposition: $R_{\hat{Q}}(G)$ splits from every compact group G. Thus, a compact group can be written as the direct product of a torsion-free group and a group with no torsion-free quotients.

Proof: Since $R^*_Q(G)$ splits from every discrete group the first statement follows by duality. The second statement is then evident from Proposition 2.5.//

We now present a result which we will use in our determination of the indecomposable compact groups.

6. Proposition: Let G be a compact connected LCA group. Then G has dense torsion subgroup iff $\hat{G}$ is reduced.

Proof: This is Proposition 7 in [A1].//

7. Corollary: If G is a compact connected LCA group then $G \approx \overline{T(G)} \times \hat{Q}^M$ where T(G) is the torsion subgroup of G and M is a cardinal number.

Proof: If G is compact and connected then $\hat{G}$ is discrete and torsion-free. Thus $\hat{G} \approx A \times (\underset{M}{\oplus} Q)$ where A is reduced and M is some cardinal number [F, 21.3 and 23.1]. Therefore $G \approx \hat{A} \times \hat{Q}^M$. By Proposition 6, $\hat{A} = \overline{T(\hat{A})}$. Since $\hat{Q}$ is torsion-free, $T(G) = T(\hat{A})$, proving that $G \approx \overline{T(G)} \times \hat{Q}^M$.//

We can now characterize the compact indecomposable groups.

8. Proposition: If G is a compact indecomposable group then G is $Z(p^n)$, J_p, $\hat{Q}$ or a compact connected torsion dense group.

Proof: If G is a compact indecomposable group then $\hat{G}$ is a discrete indecomposable group. By (i) and (ii), G is $Z(p^n)$, $Z(p^\infty)$, Q or a reduced torsion-free group, since a divisible torsion-free group is a direct sum of Q's [F, 23.1]. Taking duals, we have proved all but the last case. If $\hat{G}$ is a reduced torsion-free discrete group then G is compact and, by [H&R, 24.25], connected. Finally, by Proposition 6, G has dense torsion subgroup.//

Note that the determination of those compact connected torsion dense groups which are indecomposable is equivalent to the determination of those discrete reduced torsion-free groups which are indecomposable, a problem that is known to be quite difficult [K, remark following Theorem 10].

9. Remark: It is not difficult to determine by inspection, using the results of part two, that Conjectures 3 and 4 are satisfied by discrete groups and compact groups.

We now present several known splitting results and structure theorems in preparation for our study of indecomposable LCA groups. First, a word about terminology. We say that an LCA group H splits from an LCA group G if any closed subgroup of G topologically isomorphic to H splits from G.

10. Proposition: The identity component, $R^*_{\mathbb{R}}(G)$, splits in any torsion-free LCA group G. The subgroup of compact elements, $R_{\mathbb{R}}(G)$, splits in any densely divisible LCA group G.

Proof: The first statement is proved in [H&R, 25.30.c]. The second statement follows from the first by duality since torsion-free and densely divisible are dual by [A1, Theorem 1].//

11. Proposition: $\bigoplus_{N} Z(p^{\infty})$ can be split from any totally disconnected group. F_p can be split from any torsion-free group. $\bigoplus_{N} Q$ can be split from any group with compact identity component.

Proof: These are proved in [R2, 4.13, 4.21 and 4.23].//

The structure theorem due to Robertson which we quoted as Theorem 2.32 is a simple application of the preceding proposition since any group can be written as $\mathbb{R}^n \times G_0$ where G_0 has compact identity component [H&R, 24.30 and 7.8].

12. Proposition: If the LCA group G has a closed subgroup topologically isomorphic to a discrete divisible torsion group D then $G = D_1 \times G_1$ where $D \approx D_1$.

Proof: This is contained in [AJ, Theorem II].//

Note that the subgroup D itself may not split, although some subgroup isomorphic to it will.

The next result is a very useful characterization of nonreduced LCA groups due to Robertson [R2, 6.4].

13. Proposition: If an LCA group is not reduced then it contains a closed subgroup topologically isomorphic to $\mathbb{R}$, Q, $Z(p^{\infty})$, F_p or a quotient of $\hat{Q}$.

Proof: See [R2, 6.4].//

We are now ready to work on Conjecture 3. We start with a simple

14. Lemma: If G is an indecomposable LCA group then G is topologically isomorphic to $\mathbb{R}$, Q, $\hat{Q}$ or a group of the form H as in Theorem 2.32.

Proof: This is obvious from Theorem 2.32.//

Since $\mathbb{R}$, Q and $\hat{Q}$ are all torsion-free and divisible they all satisfy Conjecture 3. Thus in investigating Conjecture 3 we may restrict attention to LCA groups G satisfying the following conditions:

(a) G is neither discrete nor compact.

(b) G has no closed subgroups topologically isomorphic to $\mathbb{R}$ or Q.

(c) G has no quotients isomorphic to $\hat{Q}$

(d) $R_{\mathbb{R}}(G)$ is open in G and $R^*_{\mathbb{R}}(G)$ is compact.

(e) $R^*_{\mathbb{R}}(G) \leq R_{\hat{Q}}(G)$ and $R^*_Q(G) \leq R_{\mathbb{R}}(G)$.

(f) $R_{\mathbb{R}}(G) = R_Q(G)$, $R^*_{\mathbb{R}}(G) = R^*_{\hat{Q}}(G)$, $R_{\hat{Q}}(G) = R_{E_o}(G)$, and $R^*_Q(G) = R^*_{E_o}(G)$.

Note that (a) follows from Remark 9. The other conditions follow from Theorem 2.32 and Lemma 2.33.

We will now use Proposition 13 to further restrict the possibilities for a nonreduced indecomposable LCA group.

15. Lemma: If G is an indecomposable LCA group then one of the following is true:

(i) G is topologically isomorphic to $\mathbb{R}$, Q, $\hat{Q}$, $Z(p^\infty)$ or F_p.

(ii) G is reduced

(iii) G satisfies conditions (a) - (f), G is not torsion-free, and G has a closed subgroup topologically isomorphic to F_p.

(iv) G satisfies conditions (b) - (f) and has a closed subgroup topologically isomorphic to a quotient of $\hat{Q}$.

<u>Proof</u>: By Lemma 14, either G is topologically isomorphic to $\mathbb{R}$, Q or $\hat{Q}$, or else G satisfies conditions (b) - (f). Therefore, suppose G satisfies (b) - (f) and is not reduced. By Proposition 13, G contains a copy of $\mathbb{R}$, Q, $Z(p^\infty)$, F_p or a quotient of $\hat{Q}$. Now, $\mathbb{R}$ and Q are ruled out by condition (b). If G contains a copy of $Z(p^\infty)$ then it splits by Proposition 12. Since G is indecomposable, $G \approx Z(p^\infty)$ which satisfies (i). Suppose G has a closed subgroup topologically isomorphic to F_p. If G is not torsion-free then (iii) is satisfied since G is clearly neither discrete nor compact. If G is torsion-free then the copy of F_p splits by Proposition 11. Since G is indecomposable, $G \approx F_p$ which satisfies (i). Finally, if G contains a quotient of $\hat{Q}$ then (iv) is satified.//

We need to prove that an indecomposable group of type (iii) or (iv) must be densely divisible in order to prove Conjecture 3.

We now prove a lemma which asserts that we can lift a splitting from a quotient back to the original group under certain conditions.

16. Lemma: Let D and B be closed subgroups of an LCA group A such that $D \cap B = 0$ and the image of D in A/B under the natural map splits. If D is σ-compact then D splits from A.

Proof: Suppose that $A/B = (D+B)/B \times A_o/B$. Then clearly $D \cap A_o = 0$. Let $x \in A$. Then $x + B = (d+B) + (a+B)$ for some $d \in D$ and $a \in A_o$. Thus $x = d + a + b$ for some $b \in B$. But $B \leq A_o$, and thus $x = d + (a+b) \in D + A_o$. This proves that $D + A_o = A$. If D is σ-compact then $A \approx D \times A_o$ by [FG, 3.2].//

Now we apply this lemma to case (iii) of Lemma 15.

17. Lemma: If G is an indecomposable LCA group satisfying case (iii) of Lemma 15 then either $G \approx F_p$ or $R_{\hat{Q}}(G)$ contains every closed subgroup of G topologically isomorphic to F_p.

Proof: Let $R = R_{\hat{Q}}(G)$ and let $D \leq G$ such that $D \approx F_p$. Let $f: G \to G/R$ be the natural map. Since R contains the identity component of G (condition e), G/R is totally disconnected. Suppose $D \cap R = 0$. Then $f|_D$ is 1-1 so by [R2, 4.11] $f|_D$ is a topological isomorphism. Thus $f(D) \approx F_p$. Since G/R is torsion-free (Proposition 5), f(D) splits. By Lemma 16, D splits since F_p is σ-compact [H&R, 10.5]. Since G is indecomposable, $G \approx F_p$. Now suppose that $D \cap R \neq 0$. If D is not contained in R then f(D) is algebraically isomorphic to $Z(p^\infty)$ which is impossible since G/R is torsion-free. Thus $D \leq R$.//

This is the extent of our knowledge about Conjecture 3. If it is false then it is a group satisfying case

(iii) or (iv) of Lemma 15 with the additional condition from Lemma 17 in case (iii) for which it fails.

<u>Problem 3</u>: Prove or disprove Conjectures 3 and 4.

3.3 Questions

In this section we present some open problems connected with the results we have proved.

Problem 4: If G is self-dual($G \approx \hat{G}$) then $G \in \underline{S}(\hat{G})$ and $\hat{G} \in \underline{S}(G)$. Is the converse true? Characterize those groups G for which $G \in \underline{S}(\hat{G})$ and $\hat{G} \in \underline{S}(G)$. Also, if G is self-dual then $R_G = R_{\hat{G}}$. Does the converse hold? Characterize the groups G for which $R_G = R_{\hat{G}}$.

In [A3] the LCA groups G such that $G \in \underline{T}(\hat{G})$ were described. From the description obtained it was easy to show that if $G \in \underline{T}(\hat{G})$ then $\hat{G} \notin \underline{T}(G)$. Perhaps this problem should be attacked by first trying to find all LCA groups G such that $G \in \underline{S}(\hat{G})$.

Problem 5: Find necessary and sufficient conditions for a class of LCA groups to be a torsion class, a torsion-free class, a sufficiency class, or a dual sufficiency class. Stenström [S] gives necessary and sufficient conditions for a class to be a torsion class or a torsion-free class in the category of modules. His proof that his conditions are sufficient involves representing the sum of submodules as a quotient of their direct sum. This procedure does not adapt to LCA groups because the topology is not preserved properly. Also, arbitrary direct sums are not defined.

The results contained in Lemmas 1.5 and 1.8 are obvious necessary conditions.

<u>Problem 6</u>: Are there any triples ($\underline{T}_1, \underline{T}_2, \underline{T}_3$) of classes of LCA groups such that ($\underline{T}_1, \underline{T}_2$) and ($\underline{T}_2, \underline{T}_3$) are both torsion theories? Are there longer sequences of this sort?

<u>Problem 7</u>: Estimate the number of distinct idempotent radicals (if they form a set). This might be possible if necessary and sufficient conditions for a class to be a torsion or torsion-free class were known. Estimate the number of distinct idempotent radicals of the form R_H or R^*_H. These correspond to torsion theories generated or cogenerated by a single group.

<u>Problem 8</u>: Characterize left exact radicals and their associated classes.

<u>Problem 9</u>: Characterize the sufficiency classes which are closed under quotients. This is a rather restrictive condition. Clearly $\underline{S}(T)$ is one such since it contains all LCA groups. If G is not totally disconnected then it follows from [A2, 1.1 and 1.7] and [H&R, 24.12] that G has an open onto character and thus a quotient isomorphic to T. Thus $\underline{S}(G) = \underline{S}(T)$. Now, $\underline{S}(Q/Z)$, the class of all totally disconnected groups is closed under quotients. The problem then reduces to the determination of those proper subclasses of $\underline{S}(Q/Z)$ which are closed under quotients and which are sufficiency classes.

<u>Problem 10</u>: Is R_H idempotent whenever H is divisible?

Problem 11: Under what conditions on H_1 and H_2 is it true that $R_{H_1} = R_{H_2}$ or that $R_{H_1} = R^*_{H_2}$? Recall that

$R^*_{\mathbb{R}} = R_{Q/Z}$ and $R_{\mathbb{R}} = R^*_J$.

Problem 12: Study "open" radicals and coradicals defined by $O_H(G) = \bigcap_{\substack{f:G\to H \\ f \text{ open}}} \ker(f)$ and $O^*_H(G) = \overline{\sum_{\substack{f:H\to G \\ f \text{ open}}} \mathrm{im}(f)}$.

Recall that open continuous homomorphisms correspond to quotients.

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