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TWO GENERALIZATIONS OF THE ADAMS SPECTRAL SEQUENCE

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By an ordinary Adams spectral sequence with respect to a spectrum E, we shall mean the kind of spectral sequence described by Adams in [A1, Part III, Chap. 14]. Note that Adams does not assume E_*X is projective in order to construct the spectral sequence, but only to get an algebraic description of E_2 in terms of E homology. In detail, let Y be a spectrum and let

(1) $\quad Y_o \longleftarrow Y_1 \longleftarrow Y_2 \longleftarrow \cdots$

be an inverse sequence such that if we let $\overline{Y}_i = Y_i/Y_{i+1}$ then

(i) $Y \simeq Y_o$

(ii) $E_*Y_i \longrightarrow E_*\overline{Y}_i$ is a π_*E split monomorphism

and (iii) $\overline{Y}_i$ is a retract of a spectrum $\widetilde{Y}_i \wedge E$ for some $\widetilde{Y}_i$.

By applying $[X,-]_* = \pi_*F(X,-)$ to (1) we obtain an exact couple and, hence, a spectral sequence. As usual, E_1 depends on the particular resolution (1) chosen, while E_r for $r > 1$ depends only on X and Y. To indicate this dependence we adopt the following

NOTATION. Let $E_1^{st}(X,\{Y_i\}) = [X,\overline{Y}_s]_{t-s}$. For $r \geq 2$ let $E_r^{st}(X,Y) = E_r^{st}(X,\{Y_i\})$ be the usual subquotient of $E_1^{st}(X,\{Y_i\})$.

Under appropriate hypotheses this spectral sequence assumes the particularly pleasing form

$$E_2^{st} = \mathrm{Ext}_{E_*E}^{st}(E_*X,E_*Y) \Rightarrow [X,Y]_{t-s}^E ,$$

where the Ext is that of E_*E comodules relative to π_*E split exact sequences, and $[-,-]_*^E$ denotes homotopy classes of morphisms in the stable category localized at E.

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The two generalizations of the Adams spectral sequence referred to in the title are obtained by mixing (1) with either a direct or inverse sequence. (The reader is warned that we do not prove convergence here. For our applications, Propositions 8 and 9, convergence is irrelevant. See the penultimate paragraph for further remarks on convergence.) To proceed, suppose given direct and inverse sequences

(2) $X \simeq X_o \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$, and

(3) $Z \simeq Z_o \longleftarrow Z_1 \longleftarrow Z_2 \longleftarrow \cdots$.

Then $F = F(X_o,Y_o)$ and $W = Z_o \wedge Y_o$ are bifiltered by $F_{ij} = F(X_i,Y_j)$ and $W_{ij} = Z_i \wedge Y_j$. By [R2, Lemma 3.1], we may suppose each inclusion in these bifiltrations is the inclusion of a subcomplex. Then, if we totalize them by setting

$$F_n = \bigcup_{i+j=n} F_{ij} \quad \text{and} \quad W_n = \bigcup_{i+j=n} W_{ij}$$

and we let

$$\overline{X}_i = \Sigma^{-1}(X_{i+1}/X_i), \quad \overline{Z}_i = Z_i/Z_{i+1},$$

$$\overline{F}_n = F_n/F_{n+1}, \quad \text{and} \quad \overline{W}_n = W_n/W_{n+1},$$

we have the relations

$$\overline{F}_n = \bigvee_{i+j=n} F(\overline{X}_i,\overline{Y}_j) \quad \text{and} \quad \overline{W}_n = \bigvee_{i+j=n} \overline{Z}_i \wedge \overline{Y}_j .$$

Furthermore, the boundary maps $\partial : \overline{F}_n \longrightarrow \overline{F}_{n+1}$ and $\partial : \overline{W}_n \longrightarrow \overline{W}_{n+1}$ are converted by these equivalences to

$$\bigvee_{i+j=n} F(\partial,1) + F(1,\partial) \quad \text{and} \quad \bigvee_{i+j=n} \partial \wedge 1 + 1 \wedge \partial .$$

By applying $[A,-]_*$ to the sequences

$$F_o \longleftarrow F_1 \longleftarrow F_2 \longleftarrow \cdots$$

and

$$W_o \longleftarrow W_1 \longleftarrow W_2 \longleftarrow \cdots$$

we obtain spectral sequences which we shall denote by $\widetilde{E}_r$ and $\widetilde{\widetilde{E}}_r$ respectively.

The following results are standard.

PROPOSITION 1: (a) $\widetilde{E}_1^{st} = \bigoplus_i E_1^{s-i,t-i}(A \wedge \overline{X}_i, \{Y_j\})$

$$\stackrel{\approx}{E}{}_1^{st} = \bigoplus_i E_1^{s-i,t-i}(A, \overline{Z}_i \wedge \{Y_j\})$$

(b) The spectral sequences are natural with respect to A, $\{X_i\}$, $\{Y_i\}$ and $\{Z_i\}$.

(c) If they converge, then

$$\widetilde{E}_r^{st} \Rightarrow [A, F(X,Y)]^E_{t-s} = [A \wedge X, Y]^E_{t-s}$$

$$\stackrel{\approx}{E}{}_r^{st} \Rightarrow [A, Z \wedge Y]^E_{t-s} .$$

PROPOSITION 2: There are smash product pairings

$$\begin{array}{ccccc} \widetilde{E}_r(A,\{F(X_i,Y_j)\}) & \otimes & \widetilde{E}_r(A',\{F(X'_k,Y'_\ell)\}) & \longrightarrow & \widetilde{E}_r(A\wedge A',\{F(X_i\wedge X'_k, Y_j \wedge Y'_\ell)\}) \\ \Downarrow & & \Downarrow & & \Downarrow \\ [A,F(X,Y)]^E & \otimes & [A',F(X',Y')]^E & \longrightarrow & [A\wedge A', F(X\wedge X', Y\wedge Y')]^E \end{array}$$

and

$$\begin{array}{ccccc} \stackrel{\approx}{E}_r(A,\{Z_i\wedge Y_j\}) & \otimes & \stackrel{\approx}{E}_r(A',\{Z'_k\wedge Y'_\ell\}) & \longrightarrow & \stackrel{\approx}{E}_r(A\wedge A',\{Z_i\wedge Z'_k\wedge Y_j\wedge Y'_\ell\}) \\ \Downarrow & & \Downarrow & & \Downarrow \\ [A,Z\wedge Y]^E & \otimes & [A',Z'\wedge Y']^E & \longrightarrow & [A\wedge A', Z\wedge Z'\wedge Y\wedge Y']^E \end{array}$$

Proposition 2 implies that $\widetilde{E}_r$ and $\stackrel{\approx}{E}_r$ are modules over the spectral sequence $\mathrm{Ext}_{E_*E}(\pi_*E, \pi_*E) \Rightarrow \pi_*S$.

Note that if we use trivial direct and inverse systems

$$X \longrightarrow * \longrightarrow * \longrightarrow \cdots \quad \text{or} \quad Z \longleftarrow * \longleftarrow * \longleftarrow \cdots$$

then $\widetilde{E}_r$ and $\stackrel{\approx}{E}_r$ become ordinary Adams spectral sequences.

A spectral sequence is most useful when one can identify its starting point. Toward this end we introduce the following algebraic lemma, which we learned from Doug Ravenel.

Let the sequence of R-modules (R commutative with 1)

$$0 \longleftarrow P \longleftarrow P_o \longleftarrow P_1 \longleftarrow \cdots$$

$$0 \longrightarrow I \longrightarrow I_o \longrightarrow I_1 \longrightarrow \cdots$$

and

$$0 \longleftarrow Q \longleftarrow Q_o \longleftarrow Q_1 \longleftarrow \cdots$$

be exact, and let P_*, I_* and Q_* denote the complexes with the first term (P,I, or Q) omitted. Call a complex projective, injective, or flat if each term is. If $C = \{C_{ij}\}$ is a bicomplex, let Tot(C) be the complex with

$$\mathrm{Tot}(C)_n = \bigoplus_{i+j=n} C_{ij} .$$

LEMMA 3. If P_* is projective or I_* is injective then

$$H(\mathrm{Tot}(\mathrm{Hom}(P_*, I_*))) = \mathrm{Ext}(P,I) .$$

If either P_* or Q_* is flat then

$$H(\mathrm{Tot}(P_* \otimes Q_*)) = \mathrm{Tor}(P,Q) .$$

PROOF. In each case the result follows from the collapse of an appropriate bicomplex spectral sequence.//

To apply this lemma to the calculation of $\widetilde{E}_2$ and $\approx{E}_2$ we associate complexes P_*, I_*, and J_* to the sequences (1) - (3) by setting

$$I = E_*Y \qquad I_i = E_*\Sigma^i \overline{Y}_i$$

$$J = E_*Z \qquad J_i = E_*\Sigma^i \overline{Z}_i$$

$$P = E_*X \qquad P_i = E_*\Sigma^{-i} \overline{X}_i ,$$

with differentials defined in the usual way from the sequences (1) - (3).

Let us call a complex <u>trivial</u> if each differential in it is 0 .

LEMMA 4. (a) If each map in a sequence (1), (2) or (3) induces a monomorphism in E homology, the associated complex is trivial.

(b) If each map in the sequence induces an epimorphism in E homology, the associated complex is trivial.

(c) If each map in the sequence induces the zero homomorphism in E homology, the associated complex is exact.

In order to apply these algebraic lemmas we must be able to replace [X,Y] by $\mathrm{Hom}_{E_*E}(E_*X, E_*Y)$ for appropriate X and Y . To this end we assume

(i) E_*E is π_*E projective

(ii) if E_*A is π_*E projective then, for any Y,

$$[A, Y \wedge E] \cong \mathrm{Hom}_{E_*E}(E_*A, E_*(Y \wedge E)) ,$$

the isomorphism being given by taking the induced map in homology. Adams [A1, pp. 280 and 284ff] gives general conditions under which (ii) is satisfied, and shows they hold when E is S, HZ_p, MO, MU, MS_p, K or KO. Note that, by standard arguments, (ii) can be reduced to

(ii') if E_*A is π_*E projective then

$$E^*A \cong (E_*A)^* = \mathrm{Hom}_{\pi_*E}(E_*A, \pi_*E) .$$

In what follows, let $\otimes = \otimes_{\pi_*E}$, let $\mathrm{Hom} = \mathrm{Hom}_{E_*E}$ be the graded module of E_*E comodule homomorphisms, and let Ext^i be the i^{th} derived functor of Hom relative to the injective class generated by extended comodules. (These are injective relative to π_*E split exact sequences.) Totalization will either add the degrees of the two complexes involved (as in $\mathrm{Tot}(I_* \otimes J_*)$) or will add the homological degree to the degree of the sole complex involved (as in $\mathrm{Tot}(\mathrm{Ext}(E_*A \otimes P_*, E_*Y))$, whichever is appropriate. Internal degrees are not affected.

THEOREM 5. (1) If E_*A and P_* are π_*E projective then

$$\tilde{E}_1 = \mathrm{Tot}(\mathrm{Hom}(E_*A \otimes P_*, I_*))$$

and (a) if, in addition, P_* is trivial then

$$\tilde{E}_2 = \mathrm{Tot}(\mathrm{Ext}(E_*A \otimes P_*, E_*Y)) ,$$

while (b) if, in addition, P_* is exact then

$$\tilde{E}_2 = \mathrm{Ext}(E_*A \otimes E_*X, E_*Y) .$$

(2) If E_*A and either J_* or I_* are π_*E projective then

$$\tilde{\tilde{E}}_1 = \mathrm{Tot}(\mathrm{Hom}(E_*A, I_* \otimes J_*))$$

$$= \mathrm{Hom}(E_*A, \mathrm{Tot}(I_* \otimes J_*))$$

and (a) if, in addition, J_* is trivial then

$$\tilde{\tilde{E}}_2 = \mathrm{Tot}(\mathrm{Ext}(E_*A, J_* \otimes E_*Y))$$

while (b) if, in addition, J_* is π_*E split exact then

$$\tilde{\tilde{E}}_2 = \mathrm{Ext}(E_*A, E_*Z \otimes E_*Y) .$$

PROOF. Since E_*A and $E_*\overline{X}_i$ are π_*E projective in part (1), there is a Kunneth isomorphism $E_*(A \wedge \overline{X}_i) \cong E_*A \otimes E_*\overline{X}_i$, which implies that $E_*(A \wedge \overline{X}_i)$ is also π_*E projective. Therefore,

$$\begin{aligned}[A,F(\overline{X}_i,\overline{Y}_j)] &\cong [A \wedge \overline{X}_i,\overline{Y}_j] \\ &\cong \mathrm{Hom}(E_*A \otimes E_*\overline{X}_i, E_*\overline{Y}_j) .\end{aligned}$$

It follows that

$$\begin{aligned}\tilde{E}_1^{st} &= \bigoplus_{i+j=s} [A,F(\overline{X}_i,\overline{Y}_j)]_{t-s} \\ &\cong \mathrm{Hom}^{t-s}(E_*A \otimes E_*\overline{X}_i, E_*\overline{Y}_j) \\ &\cong \mathrm{Hom}^t(E_*A \otimes P_i, I_j) ,\end{aligned}$$

showing that $\tilde{E}_1 = \mathrm{Tot}(\mathrm{Hom}(E_*A \otimes P_*, I_*))$. If P_* is trivial then $d_1 = \mathrm{Hom}(1,d)$ and hence $\tilde{E}_2 = \mathrm{Tot}(\mathrm{Ext}(E_*A \otimes P_*, E_*Y))$. If P_* is exact then Lemma 3 implies that $\tilde{E}_2 = \mathrm{Ext}(E_*A \otimes E_*X, E_*Y)$.

In part (2), first note that since $\overline{Y}_j$ is a retract of some $\tilde{Y}_j \wedge E$, $\overline{Z}_i \wedge \overline{Y}_j$ is a retract of $\overline{Z}_i \wedge \tilde{Y}_j \wedge E$. Thus $[A,\overline{Z}_i \wedge \overline{Y}_j] \cong$ $\mathrm{Hom}(E_*A, E_*(\overline{Z}_i \wedge \overline{Y}_j)) \cong \mathrm{Hom}(E_*A, E_*\overline{Z}_i \otimes E_*\overline{Y}_j)$. Therefore

$$\begin{aligned}\tilde{\tilde{E}}_1^{st} &= \bigoplus_{i+j=s} [A,\overline{Z}_i \wedge \overline{Y}_j]_{t-s} \\ &\cong \bigoplus_{i+j=s} \mathrm{Hom}^{t-s}(E_*A, E_*\overline{Z}_i \otimes E_*\overline{Y}_j) \\ &\cong \bigoplus_{i+j=s} \mathrm{Hom}^t(E_*A, J_i \otimes I_j) ,\end{aligned}$$

showing that $\tilde{\tilde{E}}_1 = \mathrm{Tot}(\mathrm{Hom}(E_*A, J_* \otimes I_*)) = \mathrm{Hom}(E_*A, \mathrm{Tot}(J_* \otimes I_*))$. If J_* is trivial $\tilde{\tilde{E}}_2 = \bigoplus_i H(\mathrm{Hom}(E_*A, J_i \otimes I_*))$ and, since $J_i \otimes I_*$ is a π_*E split injective resolution of $J_i \otimes E_*Y$,

$$\tilde{\tilde{E}}_2 = \bigoplus_i \operatorname{Ext}(E_*A, J_i \otimes E_*Y) = \operatorname{Tot}(\operatorname{Ext}(E_*A, J_* \otimes E_*Y)) .$$

If J_* is π_*E split, the bicomplex spectral sequence

$$H_{II} H_I \operatorname{Hom}(E_*A, J_* \otimes I_*) \Rightarrow H(\operatorname{Tot}(\operatorname{Hom}(E_*A, J_* \otimes I_*))) = \tilde{\tilde{E}}_2$$

collapses to an isomorphism $\tilde{\tilde{E}}_2 \cong \operatorname{Ext}(E_*A, E_*Z \otimes E_*Y)$ if we let $d_I = \operatorname{Hom}(1, d \otimes 1)$ and $d_{II} = \operatorname{Hom}(1, 1 \otimes d)$. Note that because I_* is injective, we need not assume J_* is. //

COROLLARY 6. There is a spectral sequence

$$\operatorname{Ext}^{st}_{E_*E}(E_*X, E_*Y) \Rightarrow [X,Y]^E_{t-s} .$$

PROOF. The difference between this and the ordinary Adams spectral sequence is that we have not assumed E_*X is π_*E projective. Instead, construct a direct sequence $\{X_i\}$ for which P_* is a π_*E projective resolution of E_*X , as in [A1,Thm.13.6]. Then Theorem 5.1.(b) shows that, with $A = S$, the spectral sequence $\tilde{E}_r$ has the desired form. In fact we get a bit more. The map of direct systems

$$\begin{array}{ccccccc} X & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ \| & & \downarrow & & \downarrow & & \\ X & \longrightarrow & * & \longrightarrow & * & \longrightarrow & \cdots \end{array}$$

induces a homomorphism from the ordinary Adams spectral sequence into $\tilde{E}_r$ converging to $1 : [X,Y]^E \to [X,Y]^E$. If E_*X is π_*E projective this is an isomorphism from E_2 on. In general, it shows that we get the "right" E_2 term at the cost of allowing elements to have higher filtration.//

This Corollary is by no means new. In particular, it was known to Frank Adams by 1968 or 1969 [private communication]. He summarizes the method of construction by the appealing slogan (paraphrased): resolve E_*X by π_*E projectives, resolve E_*Y by (relative) E_*E injectives, and mix the resolutions geometrically. These ideas can be found in [A2, pp.50 and 54].

When E_*X is π_*E projective, the isomorphism between the two spectral sequences (from E_2 on) has the following useful consequence. If we wish to construct a homomorphism

$$\tilde{E}_r(\{X_i'\},\{Y_j\}) \longrightarrow E_r(X,Y)$$

converging to the homomorphism $f^* : [X',Y] \longrightarrow [X,Y]$ induced by a map $f : X \longrightarrow X'$, it suffices to construct a map of direct systems

$$\begin{array}{ccccccc} X & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ f\downarrow & & \downarrow & & \downarrow & & \\ X' & \longrightarrow & X_1' & \longrightarrow & X_2' & \longrightarrow & \cdots \end{array}$$

for <u>any</u> direct system $\{X_i\}$ whose associated complex P_* is exact. Roughly, instead of factoring $X \longrightarrow X' \longrightarrow X_i'$ through the 0 map, it suffices to factor it through the composite of i maps, each of which induces the 0 homomorphism in E homology. Ravenel uses this feature of $\tilde{E}_r$ very effectively in his proof of the Segal conjecture for Z_{p^n} [R1]. This is the dual of a result which Milgram [M2,M3] and the author [B1,B2] have used to identify differentials in the Adams spectral sequence for π_*S or, more generally, π_*Y for any H_∞ ring spectrum Y. We state that result in the following proposition. Let

$$S \longleftarrow S_1 \longleftarrow S_2 \longleftarrow \cdots$$

be an Adams resolution of S.

COROLLARY 7. A map $\{f_i\} : \{Z_i\} \longrightarrow \{Y_i\}$ from an inverse sequence to an Adams resolution induces a homomorphism

$$\approx{E}_r(A,\{Z_i \wedge S_j\}) \longrightarrow E_r(A,Y)$$

$$[A,Z]^E \xrightarrow{(f_0)_*} [A,Y]^E$$

PROOF. This follows, by naturality of $\overset{\approx}{E}_r$ from the isomorphism (for $r \geq 2$) $E_r(A,\{Y_i\}) \longrightarrow \overset{\approx}{E}_r(A,\{Y_i \wedge S_j\})$ induced by the maps $Y_n \simeq Y_n \wedge S \subset \bigcup_{i+j=n} Y_i \wedge S_j$. The isomorphism comes from the fact that the complexes associated to $\{Y_i\}$ and $\{Y_i \wedge S_j\}$ are both injective resolutions of E_*Y and the map between them induces a chain homomorphism covering the identity of Y . //

We wish to apply this to find families of differentials in the BP Adams spectral sequence for the homotopy of an H_∞ ring spectrum (for example, the sphere spectrum). By a construction to appear in [B2] and similar to that in [M1, §11], there are Steenrod operations

$$\beta\mathcal{P}^j : \mathrm{Ext}^{s,t} \longrightarrow \mathrm{Ext}^{sp-[(2j-t+s)(p-1)-1],tp}$$

$$\text{for} \quad t-s<2j\le t$$

if $\mathrm{Ext} = \mathrm{Ext}_{BP_*BP}(C,A)$ where C is a coalgebra and A is an algebra in the category of BP_*BP comodules. If $C = BP_*S$ and $A = BP_*Y$, where Y is an H_∞ ring spectrum, we can construct a map representing $\beta\mathcal{P}^j x$, given a map representing x . If L^n is the n-skeleton of the usual p-localization of $\Sigma^\infty(B\Sigma_p)$ [A3], and L^n_m is the quotient L^n/L^{m-1} , then for any $x \in \mathrm{Ext}^{s,n+s}(BP_*S,BP_*Y)$ we obtain a map of inverse sequences

$$(4)\qquad \begin{array}{ccccccc} \cdots \longleftarrow & \Sigma^n L^{n(p-1)+i}_{n(p-1)} & \longleftarrow \cdots \longleftarrow & \Sigma^n L^{n(p-1)}_{n(p-1)} \\ & \downarrow & & \downarrow \\ Y \longleftarrow \cdots \longleftarrow & Y_{ps-i} & \longleftarrow \cdots \longleftarrow & Y_{ps} \end{array} .$$

An appropriate multiple of the characteristic map of the $n+2j(p-1)-1$ cell represents $\beta\mathcal{P}^j x$ [B2]. Thus, if $\{Z_i\}$ is the skeletal filtration of $\Sigma^n L^{(n+s)(p-1)}_{n(p-1)}$ as in (4), Corollary 7 implies that differentials in $\tilde{\tilde{E}}_r(S,\{Z_i\})$ translate into differentials in $E_r(S,Y)$ relating the $\beta\mathcal{P}^j x$. This process of translation is begun in the following two results.

PROPOSITION 8. $\tilde{\tilde{E}}_2(S,\{Z_i\})$ contains elements $\beta\mathcal{P}^j$ in filtration $sp-[(2j-n)(p-1)-1]$ and stem degree $n+2j(p-1)-1$ for each j such that $n < 2j \le n+s$. Each $\beta\mathcal{P}^j$ generates a copy of $E_2(S,S \cup_p e^1)$. In addition, if $n=2k$ there is a generator $\mathcal{P}^k$ in filtration sp and stem degree np which generates a copy of $E_2(S,S)$.

PROOF. Since L has a cell in each nonnegative dimension congruent to 0 or $-1 \bmod 2(p-1)$, it is easy to check that the complex J_* associated to $\{Z_i\}$ contains BP_*S in each degree $sp-(2j-n)(p-1)+\epsilon$, $\epsilon = 0$ or 1, in the range sp to s (or $s-1$). (We truncate $\{Z_i\}$ below filtration s or $s-1$ since the Steenrod operations are trivial below filtration s. We truncate at $s-1$ rather than s if this is necessary to complete a pair of copies of BP_*S in adjacent degrees.) It is well-known that the differential in J_* is simply multiplication by p. Thus, J_* consists of segments

$$0 \longrightarrow BP_*S \xrightarrow{p} BP_*S \longrightarrow 0 \tag{5}$$

surrounded by 0's, together with a single copy of BP_*S in degree ps when n is even. Thus we need only show that if J'_* is the sequence in (5) then

$$H(\mathrm{Tot}(\mathrm{Hom}(BP_*S, J'_* \otimes I_*))) = \mathrm{Ext}(BP_*S, BP_*(S \cup_p e'))$$

generated by the second BP_*S in (5). This follows easily from the bicomplex spectral sequence whose first differential is $d \otimes 1$. //

Let $\doteq$ denote equality up to multiplication by a nonzero constant.

PROPOSITION 9. $d_{2p-1}\beta\theta^{j+1}X \doteq \alpha_1\beta\theta^j X$ if $j \not\equiv -1$ (p).

PROOF. The $2(j+1)(p-1)-1$ cell is attached by $-(j+1)\alpha_1$ to the $2j(p-1)-1$ cell, where α_1 is the Hopf map in $\pi_{2p-3}S$. This implies the differential $d_{2p-1}\beta\theta^{j+1} \doteq \alpha_1\beta\theta^j$ in $\tilde{\tilde{E}}_r$. By Corollary 7, it applies to $E_r(S,Y)$. //

Our final result shows that the duality between $\tilde{E}_r$ and $\tilde{\tilde{E}}_r$ becomes an isomorphism with appropriate finiteness restrictions. Let

$$Z \simeq Z_0 \longleftarrow Z_1 \longleftarrow \cdots$$

be a sequence of finite complexes, and let

$$S \longleftarrow S_1 \longleftarrow S_2 \longleftarrow \cdots$$

be an Adams resolution of S. Then we obtain

$$\tilde{\tilde{E}}_r(S,\{Z_i \wedge S_j\}) \Rightarrow \pi_* Z .$$

Applying the functor $D(X) = F(X,S)$ to $\{Z_i\}$ gives

$$DZ_o \longrightarrow DZ_1 \longrightarrow \cdots$$

from which we may construct

$$\tilde{E}_r(S,\{F(DZ_i,S_j)\}) \Rightarrow \pi_* DDZ .$$

In general, the maps

$$Z_i \wedge S_j \longrightarrow F(DZ_i,S_j)$$

adjoint to evaluation $DZ_i \wedge Z_i \wedge S_j \longrightarrow S_j$ induce a homomorphism $\tilde{\tilde{E}}_r \longrightarrow \tilde{E}_r$ which converges to the homomorphism induced by the canonical map $Z \longrightarrow DDZ$. Since all the Z_i are finite complexes, the maps

$$Z_i \wedge S_j \longrightarrow F(DZ_i,S_j)$$

are all equivalences, which means that the two spectral sequences are isomorphic. In fact, the same argument shows the following

PROPOSITION 10. If $\{Z_i\}$ is an inverse sequence of finite complexes, there are isomorphisms

$$\begin{array}{ccc} \tilde{\tilde{E}}_r(A,\{Z_i \wedge Y_j\}) & \xrightarrow{\cong} & \tilde{E}_r(A,\{F(DZ_i,Y_j)\}) \\ \Downarrow & & \Downarrow \\ [A,Z \wedge Y]^E & \xrightarrow{\cong} & [A,F(DZ,Y)]^E \end{array}$$

which make this square commute.

The problem of convergence for $\tilde{E}_r$ and $\tilde{\tilde{E}}_r$ seems somewhat more difficult than for the ordinary Adams spectral sequence. In [B2] it is shown that

$$\tilde{\tilde{E}}_r(A,\{Z_i \wedge S_j\}) \Rightarrow [A,Z]^E$$

converges if $\underleftarrow{h\lim}\, Z_i \simeq *$, if the complex J_* associated to $\{Z_i\}$ is trivial, and the conditions for convergence of the ordinary Adams spectral sequence

$$E_r(A,Z) \Rightarrow [A,Z]^E$$

are met [A1, §15]. In general the most optimistic conjecture is that $\tilde{E}_r(A,\{F(X_i,Y_j)\})$ converges to $[A \wedge F,Y]^E$ and $\tilde{\tilde{E}}_r(A,\{Z_i \wedge Y_j\})$ converges to $[A,C \wedge Y]^E$, where

$$F = \text{Fiber}(X \longrightarrow \underrightarrow{\text{h lim}}\, X_i)$$

and $$C = \text{Cofiber}(\underleftarrow{\text{h lim}}\, Z_i \longrightarrow Z)\ ,$$

under the conditions which would ensure convergence of the ordinary Adams spectral sequences abutting to these groups.

To conclude, let us indicate what is new here. The construction of $\tilde{E}_r$ was known to Frank Adams in 1968 or 1969, though I learned it from Ravenel's paper [R1]. The identification of $\tilde{E}_2$ in Theorem 5 I learned from Doug Ravenel. The special case of $\tilde{\tilde{E}}_r$ in which $\{Z_i\}$ is the filtration of a finite complex by skeleta and the associated complex J_* is trivial was used by Milgram in [M2]. (He also assumed $E = HZ_p$.) Another special case of $\tilde{\tilde{E}}_r$, in which $\{Z_i\}$ is an Adams resolution for a different ring theory, was used by Haynes Miller [M4]. Thus, it is the construction of $\tilde{\tilde{E}}_r$ in general and the observation that all these spectral sequences come from two dual constructions which is new here. This includes all the results about $\tilde{\tilde{E}}_r$, essentially 5.b and 7-10.

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