The Root Invariant

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Outline

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History

$G = C_2$
ξ = nontrivial 1-dim real representation of G
$S^{k\xi+n}$ = one point compactification of $k\xi + n$, so $(S^{k\xi+n})^G = S^n$.

The *Bredon-Löffler conjecture* concerns the f.p. hom $\phi_k(f) = f^G$,

$$\phi_k : [S^{k\xi}, S^0]^G_n \longrightarrow [S^0, S^0]_n,$$

and the associated *Bredon filtration*

$$F_k = \mathrm{im}(\phi_k)$$

Clearly $F_k \supset F_{k+1}$.

Bredon conjectured (1967) and Landweber proved (1969) that, in π_0, this is closely related to the vector fields number

$$v(k) = |\{i \mid 0 < i < k \text{ and } k \equiv 0, 1, 2, 4 \pmod 8\}|$$

Theorem

$$F_k\pi_0 = \begin{cases} 2^{v(k)+2} & k \equiv 0 \pmod 4 \\ 2^{v(k)+1} & k \not\equiv 0 \pmod 4 \end{cases}$$

Bredon also made the following elementary observation.

Lemma (Bredon)

$\pi_n(S^0) = F_0 = F_1 = \cdots = F_n$.

Proof.

$$(S^n \xrightarrow{g} S^0) = \phi_n(S^{n\xi+n} \simeq S^n \wedge S^n \xrightarrow{g \wedge g} S^0 \wedge S^0 \simeq S^0)$$

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Examples

Consider $\mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$ with usual G actions, so that the f.p. are $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$.
The G-equivariant Hopf map

$$\begin{array}{ccc} S(\mathbb{C}^2) & \xrightarrow{\widetilde{\eta}} & \mathbb{CP}^1 \\ \| & & \| \\ S^{2\xi+1} & \longrightarrow & S^{\xi+1} \end{array} \qquad \text{is in} \qquad [S^{\xi}, S^0]_0^G$$

and $\phi_1(\widetilde{\eta}) = 2$, the non-equivariant real Hopf map

$$\begin{array}{ccc} S(\mathbb{R}^2) & \longrightarrow & \mathbb{RP}^1 \\ \| & & \| \\ S^1 & \xrightarrow{2} & S^1 \end{array} \qquad \text{in} \qquad [S^0, S^0]_0.$$

Similarly, the G-equivariant quaternionic Hopf map

$$\begin{array}{ccc} S(\mathbb{H}^2) & \xrightarrow{\widetilde{\nu}} & \mathbb{HP}^1 \\ \| & & \| \\ S^{4\xi+3} & \longrightarrow & S^{2\xi+2} \end{array} \qquad \text{is in} \qquad [S^{2\xi}, S^0]_1^G$$

and $\phi_2(\widetilde{\nu}) = \eta$, the non-equivariant complex Hopf map

$$\begin{array}{ccc} S(\mathbb{C}^2) & \longrightarrow & \mathbb{CP}^1 \\ \| & & \| \\ S^3 & \xrightarrow{\eta} & S^2 \end{array} \qquad \text{in} \qquad [S^0, S^0]_1.$$

Similarly $\phi_4(\widetilde{\sigma}) = \nu$.

We find, taking composites, that

$$\pi_1 = F_2 \supset F_3 = 0$$

$$\pi_2 = F_4 \supset F_5 = 0$$

and

$$\begin{array}{ccccccccc} \pi_3 & \xleftarrow{=} & F_4 & \xleftarrow{\supset} & F_5 & \xleftarrow{\supset} & F_6 & \xleftarrow{\supset} & F_7 \\ & & \| & & \| & & \| & & \| \\ & & \langle \nu \rangle & & \langle 2\nu \rangle & & \langle 4\nu \rangle & & 0 \end{array}$$

Bredon-Löffler Conjecture

Conjecture (Bredon-Löffler)

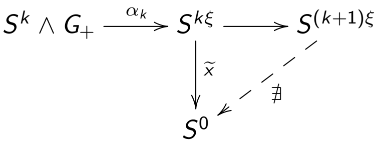
If $n > 0$ then $F_{2n+1}\pi_n = 0$.
That is, if $k > 2n > 0$ then the image of

$$\phi_k : [S^{k\xi}, S^0]^G_n \longrightarrow [S^0, S^0]_n,$$

is zero.

The Bredon Root Invariant

Suppose that $x \in F_k\pi_n \setminus F_{k+1}\pi_n$. Then there are $\widetilde{x} : S^{k\xi} \longrightarrow S^0$ with $\phi_k(\widetilde{x}) = x$ but no such $\widetilde{x}$ extends to $S^{(k+1)\xi}$.



The adjoint of $\widetilde{x}\alpha_k$ is the underlying non-equivariant map

$$U_k(\widetilde{x}) \in [S^k, S^0]_n = \pi_{n+k}(S^0).$$

Definition

If $x \in F_k\pi_n \setminus F_{k+1}\pi_n$ then the *Bredon root invariant* $B(x)$ is the coset $U_k(\phi_k^{-1}(x)) \subset \pi_{n+k}(S^0)$.

Write $|x| = n$ for $x \in \pi_n(S^0)$. The easy Lemma about the Bredon filtration can be restated in terms of the Bredon root invariant.

Corollary

$|B(x)| \geq 2|x|$.

We can similarly restate the Bredon-Löffler conjecture.

Conjecture

$|B(x)| \leq 3|x|$.

$$\begin{array}{ccccccccc}
\pi_n = F_n & \xleftarrow{\supset} & F_{n+1} & \xleftarrow{\supset} & \cdots & \xleftarrow{\supset} & F_{2n} & \xleftarrow{\supset} & F_{2n+1} = 0 \\
\downarrow B & & \downarrow B & & & & \downarrow B & & \\
\pi_{2n} & & \pi_{2n+1} & & \cdots & & \pi_{3n} & &
\end{array}$$

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 & & \downarrow B & & \downarrow B & & & & \downarrow B & & \\
 & & \pi_{2n} & & \pi_{2n+1} & & \cdots & & \pi_{3n} & &
\end{array}$$

Our examples give

- $B(2) = \eta$, $B(4) = \eta^2$, $B(8) = \eta^3 = 4\nu$
- $B(\eta) = \nu$
- $B(\nu) = \sigma$

However, we will see that $B(\sigma) = \sigma^2$.

There is also the elementary observation

Theorem (B)

- $|B(xy)| \geq |B(x)| + |B(y)|$
- *If* $|B(xy)| = |B(x)| + |B(y)|$ *then* $B(x)B(y) \subset B(xy)$.

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Non-equivariant Prehistory

Mahowald's 1967 AMS Memoir *The Metastable Homotopy of S^n*

DEFINITION 4.4. Let α be an element of either Ext or π_* for a sphere. Let $i\colon S^n \to P_n$, the inclusion onto the bottom cell. Suppose there is a j such that for $P_{n-j} \xrightarrow{p} P_n$ $i_*(\alpha) \not\in \operatorname{im} p_*$ stably. Let j be the smallest integer with this property. Then consider

$$\begin{array}{ccccc} S^{n-j} & \to & P_{n-j} & \xrightarrow{p_2} & P_{n-j+1} \\ & & & & \downarrow p_1 \\ & & & & P_n \end{array}$$

By the root of α, $\sqrt{\alpha}$ we mean $\alpha_*(\bar{\alpha})$ for any $\bar{\alpha}$ satisfying $p_1{}^*\bar{\alpha} = i^*\alpha$. Then $n-j$ is the dimension of the root.

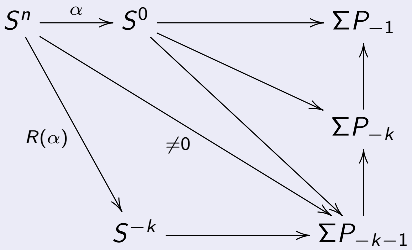
History

Let $P_k = T(k\xi)$, where ξ is the nontrivial line bundle over $\mathbb{RP}^\infty$.
Lin's Theorem tells us that $S^0 \xrightarrow{\simeq} \lim_k \Sigma P_{-k}$ and there is the associated *Mahowald filtration.*

$$M_k := \ker(\pi_n S^0 \longrightarrow \Sigma P_{-k})$$

Definition (The Mahowald Root Invariant)

For $x \in M_k \setminus M_{k+1}$, let $R(\alpha)$ be the set of lifts:



- Mahowald's work on the AHSS $\Longrightarrow \pi_*(P_n)$ gave him extensive knowledge of this filtration and the root invariants.
- Applications to the EHP-SS and its variants.
- *Metastable* homotopy of the sphere.
- Mahowald and Ravenel (Topology, 1988/1993) collected much of what was known and
  - ask for a Cartan formula for the root invariant,
  - conjecture that $|R(x)| \leq 3|x|$.
  - conjecture that if R_k is the subgroup generated by root invariants in $\pi_k(S^0)$, then

$$\lim_{k \longrightarrow \infty} \frac{\log_p \#(\pi_k(S^0))}{\log_p \#(R_k)} = \frac{1}{p^2},$$

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A 1992 email from John Greenlees

- proved that the Bredon and Mahowald filtrations agree,
- proposed the Bredon-Löffler conjecture as an interesting subject of study,
- said that Peter May had told him that my programs might be able to compute the analogous filtration at the E_2-term, and
- asked if I was interested in doing this, if so.
- They could and I was; so began an internet collaboration,
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First result:

Theorem (B&G)

- $F_k = M_k$
- $B = R$

Proof:
Obstruction theory implies

$$[X, Y \wedge S^{\infty\xi}]_n^G \cong [X^G, Y^G]_n$$

Thus, ϕ_k is induced by the inclusion $S^0 \longrightarrow S^{\infty\xi}$. The cofiber sequence $EG_+ \longrightarrow S^0 \longrightarrow S^{\infty\xi}$ gives the l.e.s

$$\cdots \longrightarrow [S^{k\xi}, S^0]_n^G \xrightarrow{\phi_k} [S^{k\xi}, S^{\infty\xi}]_n^G \longrightarrow [S^{k\xi}, \Sigma EG_+]_n^G \longrightarrow \cdots$$

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Thus,

$$\begin{aligned} F_k &= \mathrm{im}(\phi_k) \\ &= \ker([S^{k\xi}, S^{\infty\xi}]^G_n \longrightarrow [S^{k\xi}, \Sigma EG_+]^G_n \\ &= M_k \end{aligned}$$

where we use

$$\begin{aligned} [S^{k\xi}, \Sigma EG_+]^G_n &= [S^0, \Sigma EG_+ \wedge S^{-k\xi}]^G_n \\ &\cong [S^0, (\Sigma EG_+ \wedge S^{-k\xi})/G]_n \\ &= [S^0, \Sigma P_{-k}]_n \end{aligned}$$

□

To show $R(x) = B(x)$, let $x \in F_k \setminus F_{k+1}$.

Map the sequence

$$S^k \wedge G_+ \longrightarrow S^{k\xi} \longrightarrow S^{(k+1)\xi} \longrightarrow S^{k+1} \wedge G_+$$

to the sequence

$$EG_+ \longrightarrow S^0 \longrightarrow S^{\infty\xi} \longrightarrow \Sigma EG_+$$

and use the isomorphisms

- $[S^{j\xi}, S^{\infty\xi}]_n^G \cong [S^0, S^0]n$
- $[S^j \wedge G_+, X]_n^G \cong [S^j, X]_n$
- $[S^j, S^{\infty\xi}]_n = 0$

for $j = k$ and $k + 1$, to get a grid of exact sequences.

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$R(x)$ is the lift of x to the lower right corner, and $B(x)$ is the lift of x to the upper left corner.

$$
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\uparrow & & \uparrow{\scriptstyle U_k} & & \uparrow & & \uparrow \\
[S^{k\xi}, EG_+]_n^G & \longrightarrow & [S^{k\xi}, S^0]_n^G & \xrightarrow[\phi_k]{} & [S^0, S^0]_n & \longrightarrow & [S^{k\xi}, \Sigma EG_+]_n^G \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \cong} & & \uparrow \\
[S^{(k+1)\xi}, EG_+]_n^G & \longrightarrow & [S^{(k+1)\xi}, S^0]_n^G & \xrightarrow[\phi_{k+1}]{} & [S^0, S^0]_n & \longrightarrow & [S^{(k+1)\xi}, \Sigma EG_+]_n^G \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
[S^{k+1}, EG_+]_n & \xrightarrow{\cong} & [S^{k+1}, S^0]_n & \longrightarrow & 0 & \longrightarrow & [S^{k+1}, \Sigma EG_+]_n
\end{array}
$$

A standard result about maps of cofiber sequences into fiber sequences shows they agree. □

Consequences:

- Simple equivariant description of $R(X)$.
- The Cartan formula.
- Elementary proof that $|R(x)| \geq 2|X|$.
- The Mahowald-Ravenel conjecture = the Bredon-Löffler conjecture.

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Ext analog

Exactness of

$$\begin{array}{ccccc}
[S^{k\xi}, S^0]_n^G & \xrightarrow{\phi_k} & [S^{k\xi}, S^{\infty\xi}]_n^G & \longrightarrow & [S^{k\xi}, \Sigma EG_+]_n^G \\
\| & & \downarrow \cong & & \downarrow \cong \\
[S^{k\xi}, S^0]_n^G & \longrightarrow & [S^0, S^0]_n & \longrightarrow & [S^0, \Sigma P_{-k}]_n
\end{array}$$

shows

(BL Conj) $\phi_k = 0$ for $k > 2n > 0$

is equivalent to

$\pi_n(S^0) \longrightarrow \pi_n(\Sigma P_{-k})$ mono for $k > 2n > 0$.

Let $L_{-k} = H^* P_{-k}$. The map $S^0 \longrightarrow \Sigma P_{-k}$ above induces the non-zero homomorphism $r_k : \Sigma L_{-k} \longrightarrow \mathbb{F}_2$.

Conjecture (The algebraic Bredon-Löffler conjecture)

$$r_k^* : \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_A^{s,t}(\Sigma L_{-k}, \mathbb{F}_2)$$

is a monomorphism if $k > 2(t - s) > 0$.

We showed

Theorem (B&G)

The algebraic Bredon-Löffler conjecture holds for $t - s < 30$.

While preparing this talk, I checked

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- Sharp for $h_1 P^i h_1$ and $h_1^2 P^i h_1$ in the range calculated. (This is likely accessible for all i.)
- In 1996, I was able to show the much weaker bound

$$\sqrt{3 + \frac{k}{2}} > t - s + 2.$$

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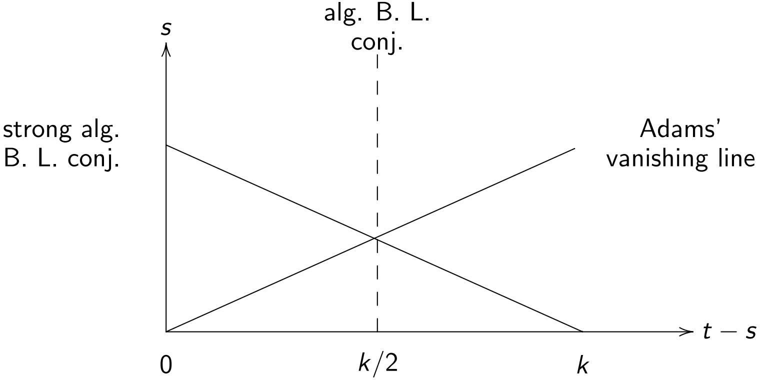
Conjecture (Strong algebraic Bredon-Löffler conjecture)

r_k^* is a monomorphism if

$$s < \frac{k-n}{2}.$$



- This is also correct in degree 0, giving Landweber's result.
- Like Adams' vanishing line, it is probably not a straight line.
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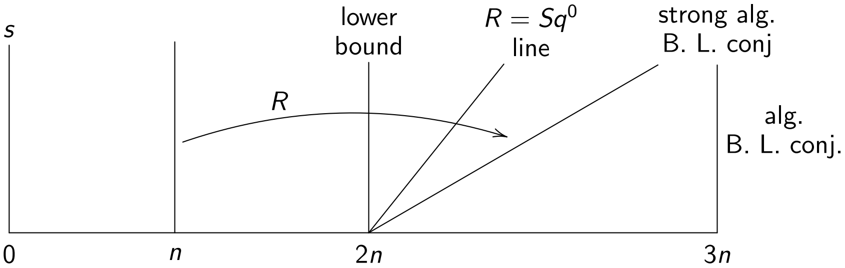

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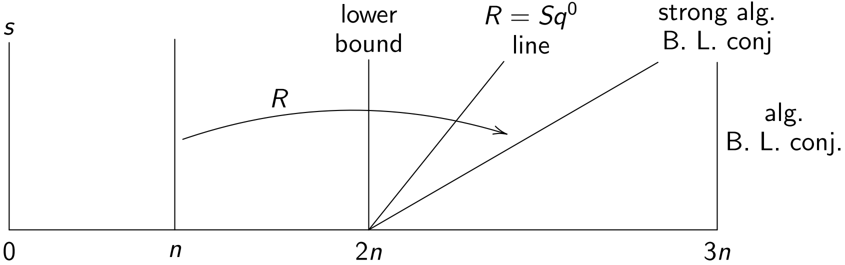

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Further work

- Mark Behrens, 'Root invariants in the Adams spectral sequence'. Trans. AMS (2006).
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- J. D. Quigley, 'The Motivic Mahowald invariant', arXiv:1801.06035 and ff.
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Outline

In recognition of the important role
that John Greenlees has played in
bringing equivariant methods into play,

Happy

First

Non-abelian

Simple

Birthday,

John