The mod 2 Adams Spectral Sequence for tmf_*

Robert Bruner

Wayne State University, and
Universitetet i Oslo

Isaac Newton Institute
11 September 2018

Outline

Acknowledgements

A report on joint work in progress with John Rognes.

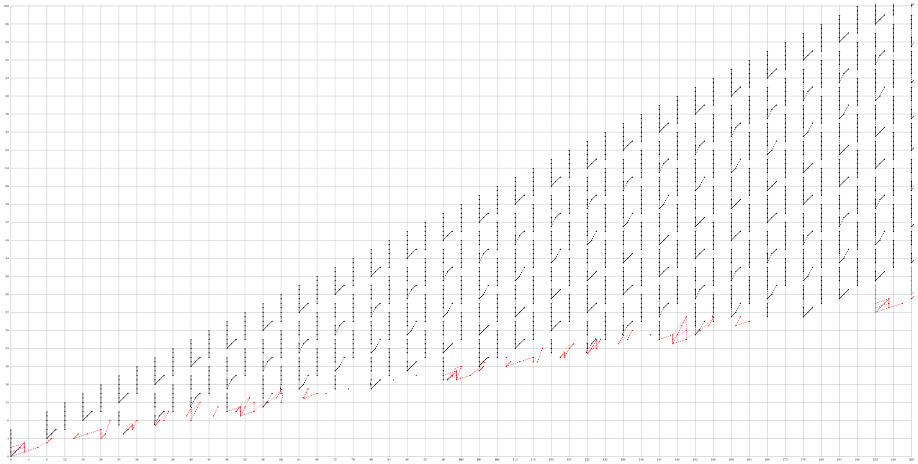
Thanks are due to the Simons Foundation for travel funds and to the Universitetet i Oslo for giving me excellent working conditions and colleagues during these last three years.

Thanks also the the Newton Institute, the organizers of the HHH programme, and the Institute staff.

Introduction

- *tmf*-modules whose ordinary homology we can readily calculate, but whose *BP* homology is much harder to get.
- The ordinary mod 2 Adams spectral sequence is thus a reasonable tool to understand these. We first needed to understand the ordinary Adams spectral sequence for *tmf* itself in greater detail.
- Have *tmf*, $tmf \wedge C2$, $tmf \wedge C\eta$, and $tmf \wedge C\nu$ in gory detail.
- Tools: DMSS, Gröbner bases, ways to make calculations finite.
- Oddity: no Toda brackets needed except as a heuristic.
- Side benefit: independent verification of earlier results, with thorough documentation.

tmf_*



- We may think of *ko* as made up of two $H\mathbb{Z}$'s with a bit of 2-torsion tagging along.
- In the same way, *tmf* is made up of eight '*ko*'s' with some B-torsion tagging along.
- Here, $B \in \pi_8(tmf)$ maps to the Bott class in $\pi_8(ko)$ under a natural map $tmf \longrightarrow ko$. We will call this lift to *tmf* the 'Bott class' as well.

Generators of tmf_*

First, there is the periodicity element $M \in \pi_{192}(tmf)$, not a zero-divisor. Group remaining generators by $\sqrt{\mathrm{Ann}_{\mathbb{Z}[B]}}$: for $0 \leq k \leq 7$,

(0) $D_k \in \pi_{24k}(tmf)$, $B_k \in \pi_{24k+8}(tmf)$, $C_k \in \pi_{24k+12}(tmf)$

(2) $\eta_k \in \pi_{24k+1}(tmf)$, and

(2, B) $\nu_k \in \pi_{24k+3}(tmf)$,
$\epsilon_k \in \pi_{24k+8}(tmf)$,
$\kappa_k \in \pi_{24k+14}(tmf)$, and
$\bar{\kappa}_k \in \pi_{24k+20}(tmf)$.

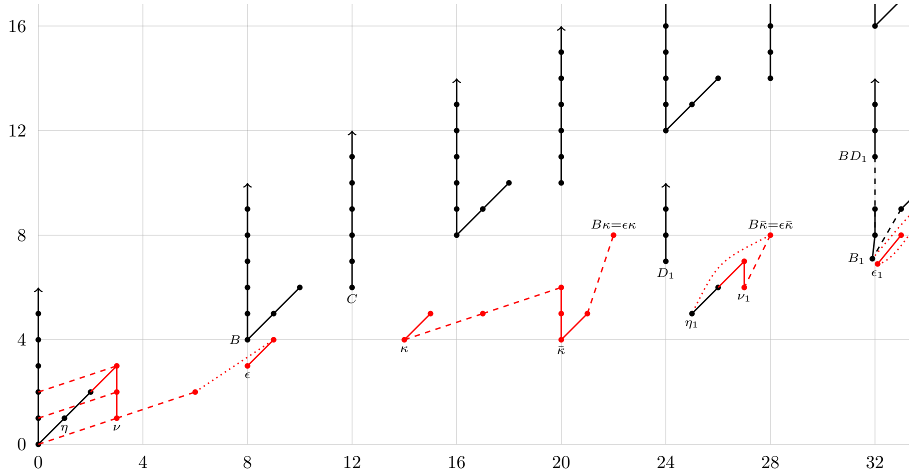
- D_k, B_k, C_k and ν_k are defined for all k.
- The rest are defined only for some values of k.

When $k = 0$, we have familiar elements:

- The unit $i : S \longrightarrow tmf$
  sends 1, η, ν, ϵ, κ and $\bar{\kappa}$
  to $D_0 = 1$, η_0, ν_0, ϵ_0, κ_0 and $\bar{\kappa}_0$.

  Omit the subscript 0 accordingly.
- The map $tmf_* \longrightarrow ko_*$ sends $D_0 = 1$, $B_0 = B$, and $C_0 = C$ to generators of ko_* in degrees 0, 8 and 12, respectively.
- The relation between B and $\bar{\kappa}$ is the same as that between 2 and η, or between η and ν, a kind of 'Frobenius', detected by Sq^0 in Ext.
- $\kappa^2 = B\bar{\kappa}$.

tmf_*


16
12
8
4
0
0
4
8
12
16
20
24
28
32
η
ν
B
ϵ
C
κ
κ̄
Bκ=ϵκ
D₁
η₁
ν₁
Bκ̄=ϵκ̄
BD₁
B₁
ϵ₁

All the generators except M:

Start+	0	8	12	1	3	8	14	20
0	1	B	C	η	ν	ϵ	κ	$\bar{\kappa}$
24	D_1	B_1	C_1	η_1	ν_1	ϵ_1		
48	D_2	B_2	C_2		ν_2			
72	D_3	B_3	C_3		(ν_3)			
96	D_4	B_4	C_4	η_4	ν_4	ϵ_4	κ_4	$(\bar{\kappa}_4)$
120	D_5	B_5	C_5		ν_5	ϵ_5		
144	D_6	B_6	C_6		ν_6			
168	D_7	B_7	C_7		(ν_7)			

Here, $\nu_3 = \eta_1^3$, $\bar{\kappa}_4 = \bar{\kappa} D_4$ and $\nu_7 = 0$ are not needed to generate $\pi_*(tmf)$, but are convenient in general formulas.

Families of elements in tmf_*

- Consider the natural map $tmf_* \longrightarrow MF_{*/2}$ to the ring of modular forms

$$MF_* = \mathbb{Z}[c_4, c_6, \Delta]/(1728\Delta = c_4^3 - c_6^2) = \mathbb{Z}[c_4, \Delta][\sqrt{c_4^3 - 12^3\Delta}]$$

- The discriminant Δ is not in the image of the map to $MF_{*/2}$, but it does exist in the spectral sequences leading to tmf_*
- Differential on Δ kills $\nu\bar{\kappa}$, detected by $h_2 g$ in the Adams spectral sequence, giving Massey products at E_2 of the Adams spectral sequence

$$\Delta(x) = \langle h_2, g, x \rangle$$

$$\Delta'(x) = \langle g, h_2, x \rangle$$

Theorem

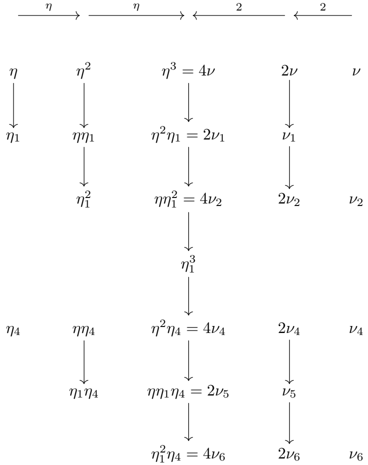
Repeated application of Δ gives classes detecting the following sequences of elements of $\pi_*(tmf)$:

- $8D \mapsto D_1 \mapsto 2D_2 \mapsto D_3 \mapsto 4D_4 \mapsto D_5 \mapsto 2D_6 \mapsto D_7 \mapsto 8M$.
- $C \mapsto C_1 \mapsto C_2 \mapsto C_3 \mapsto C_4 \mapsto C_5 \mapsto C_6 \mapsto C_7 \mapsto CM$.
- $B+\epsilon \mapsto B_1+\epsilon_1 \mapsto B_2 \mapsto B_3 \mapsto B_4+\epsilon_4 \mapsto B_5+\epsilon_5 \mapsto B_6 \mapsto B_7 \mapsto BM$.
- $8B \mapsto 8B_1 \mapsto 8B_2 \mapsto 8B_3 \mapsto 8B_4 \mapsto 8B_5 \mapsto 8B_6 \mapsto 8B_7 \mapsto 8BM$.

Δ' on h_1 and h_2 in Ext

$\xrightarrow{h_1}$	$\xrightarrow{h_1}$	$\xleftarrow{h_0}$	$\xleftarrow{h_0}$	
h_1	h_1^2	$h_1^3 = h_0^2 h_2$	$h_0 h_2$	h_2
↓	↓	↓	↓	
γ	$h_1\gamma$	$h_1^2\gamma = h_0\alpha\beta$	$\alpha\beta$	
↓	↓	↓	↓	
$h_1 w_2$	$\{h_1^2 w_2, \gamma^2\}$	$h_1\gamma^2 = h_0^2 h_2 w_2$	$h_0 h_2 w_2$	$h_2 w_2$
↓	↓	↓	↓	
γw_2	$h_1\gamma w_2$	$\{h_1^2\gamma w_2 = h_0\alpha\beta w_2, \gamma^3\}$	$\alpha\beta w_2$	
↓	↓	↓	↓	
$h_1 w_2^2$	$\{h_1^2 w_2^2, \gamma^2 w_2\}$	$h_1\gamma^2 w_2 = h_0^2 h_2 w_2^2$	$h_0 h_2 w_2^2$	$h_2 w_2^2$
↓	↓	↓	↓	
γw_2^2	$h_1\gamma w_2^2$	$\{h_1^2\gamma w_2^2 = h_0\alpha\beta w_2^2, \gamma^3 w_2\}$	$\alpha\beta w_2^2$	
↓	↓	↓	↓	
$h_1 w_2^3$	$\{h_1^2 w_2^3, \gamma^2 w_2^2\}$	$h_1\gamma^2 w_2^2 = h_0^2 h_2 w_2^3$	$h_0 h_2 w_2^3$	$h_2 w_2^3$
↓	↓	↓	↓	
γw_2^3	$h_1\gamma w_2^3$	$\{h_1^2\gamma w_2^3 = h_0\alpha\beta w_2^3, \gamma^3 w_2^2\}$	$\alpha\beta w_2^3$	
↓	↓	↓	↓	
$h_1 w_2^4$	$\{h_1^2 w_2^4, \gamma^2 w_2^3\}$	$h_1\gamma^2 w_2^3 = h_0^2 h_2 w_2^3$	$h_0 h_2 w_2^3$	$h_2 w_2^3$

Effect on η_i and ν_i



Let $\Gamma_B(tmf_*)$ be the B-power torsion, those $x \in tmf_*$ such that $B^i x = 0$ for $i >> 0$.

Theorem

$\nu_k, \epsilon_k, \kappa_k, \bar{\kappa} \in \Gamma_B(tmf_*)$.

Theorem

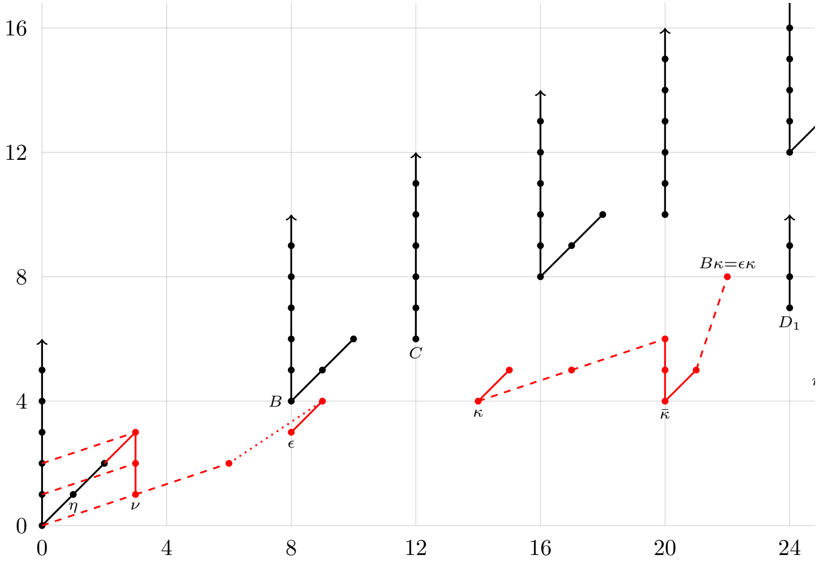
If $x \in \Gamma_B(tmf_*)$ then $Bx = \epsilon x$.

- General principle:

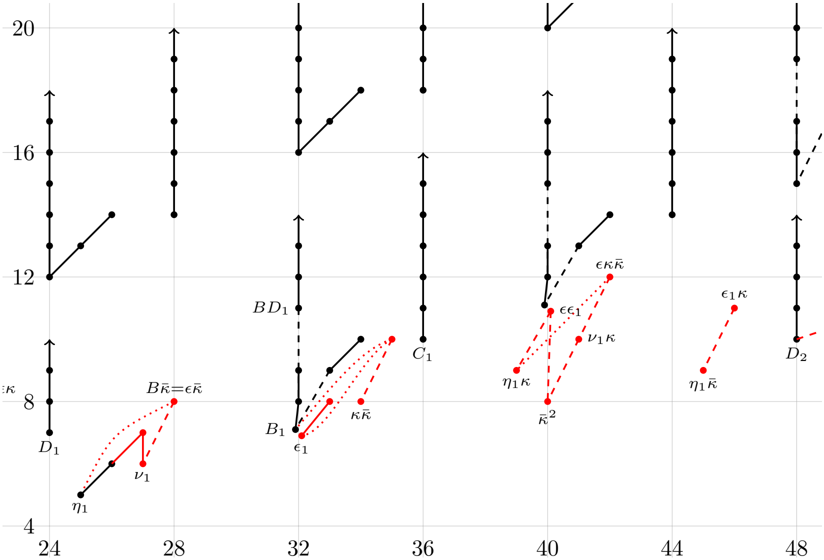
  if $x, y \in \{\eta, \nu, \epsilon, \kappa\}$ then $x_i y_j$ depends only on x, y, and $i + j$.

- Exceptions stem from the varying 2-divisibility of the $24k + 3$ stem. For example,
  - $\nu_1 \nu_k = \begin{cases} 2\nu\nu_{k+1} & k = 1, 5 \\ 0 & \text{other } k \leq 7 \end{cases}$
  - $\nu_2\nu_2 = \nu\nu_4$, $\nu_2\nu_4 = \nu\nu_6$, and $\nu_2\nu_6 = \nu\nu_8 = \nu^2 M$.
    (In general, let $x_{k+8} = x_k M$.)
  - $\nu_4\nu_6 = \pm\nu\nu_2 M$.

0 to 24



24 to 48



20
16
12
8
4
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28
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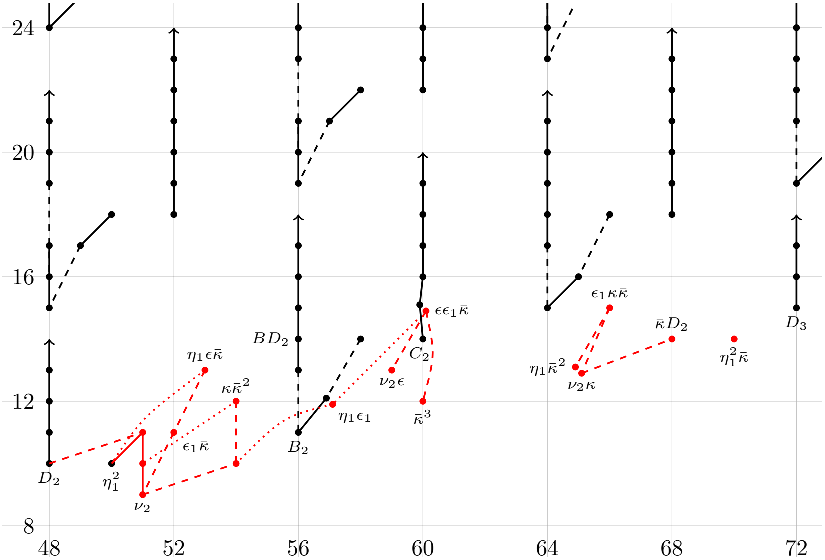
36
40
44
48
$\epsilon\kappa$
D_1
η_1
ν_1
$B\bar{\kappa}=\epsilon\bar{\kappa}$
BD_1
B_1
ϵ_1
$\kappa\bar{\kappa}$
C_1
$\eta_1\kappa$
$\epsilon\epsilon_1$
$\nu_1\kappa$
$\epsilon\kappa\bar{\kappa}$
$\bar{\kappa}^2$
$\eta_1\bar{\kappa}$
$\epsilon_1\kappa$
D_2

48 to 72



24
20
16
12
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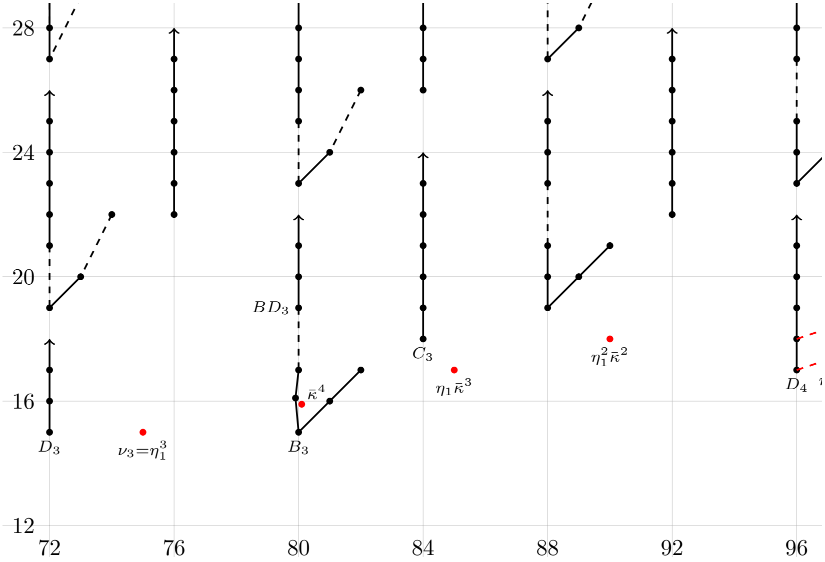
52
56
60
64
68
72
$\eta_1\epsilon\bar{\kappa}$
$\kappa\bar{\kappa}^2$
$\epsilon_1\bar{\kappa}$
D_2
η_1^2
ν_2
BD_2
B_2
$\eta_1\epsilon_1$
$\nu_2\epsilon$
C_2
$\bar{\kappa}^3$
$\epsilon\epsilon_1\bar{\kappa}$
$\epsilon_1\kappa\bar{\kappa}$
$\eta_1\bar{\kappa}^2$
$\nu_2\kappa$
$\bar{\kappa}D_2$
$\eta_1^2\bar{\kappa}$
D_3

72 to 96



28
24
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12
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76
80
84
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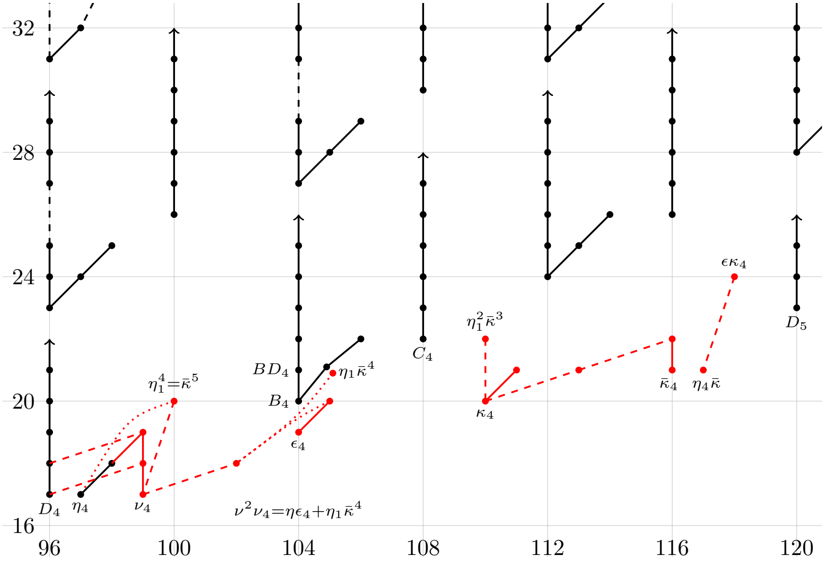
92
96
D_3
$\nu_3 = \eta_1^3$
B_3
$\bar{\kappa}^4$
BD_3
C_3
$\eta_1 \bar{\kappa}^3$
$\eta_1^2 \bar{\kappa}^2$
D_4

96 to 120


32
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108
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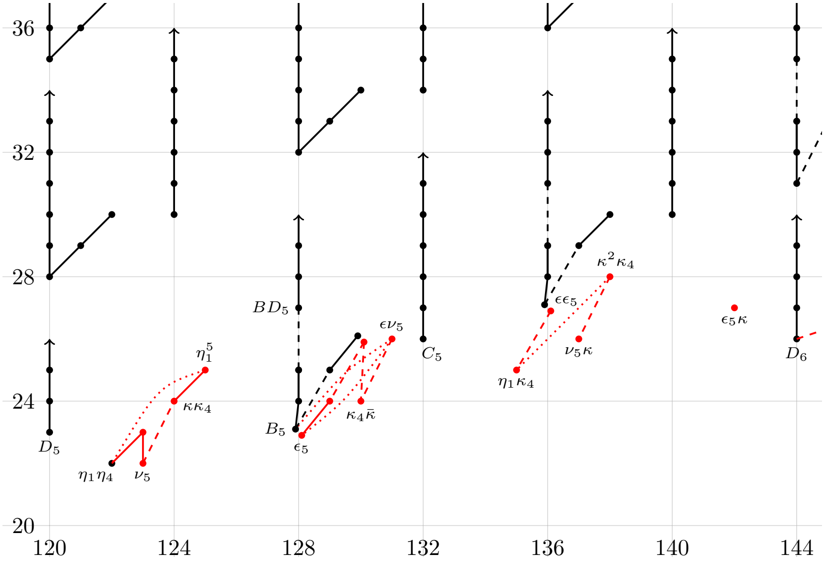
116
120
D_4
η_4
ν_4
$\eta_1^4 = \bar\kappa^5$
$\nu^2\nu_4 = \eta\epsilon_4 + \eta_1\bar\kappa^4$
BD_4
B_4
$\eta_1\bar\kappa^4$
ϵ_4
C_4
$\eta_1^2\bar\kappa^3$
κ_4
$\bar\kappa_4$
$\eta_4\bar\kappa$
$\epsilon\kappa_4$
D_5

120 to 144



36
32
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24
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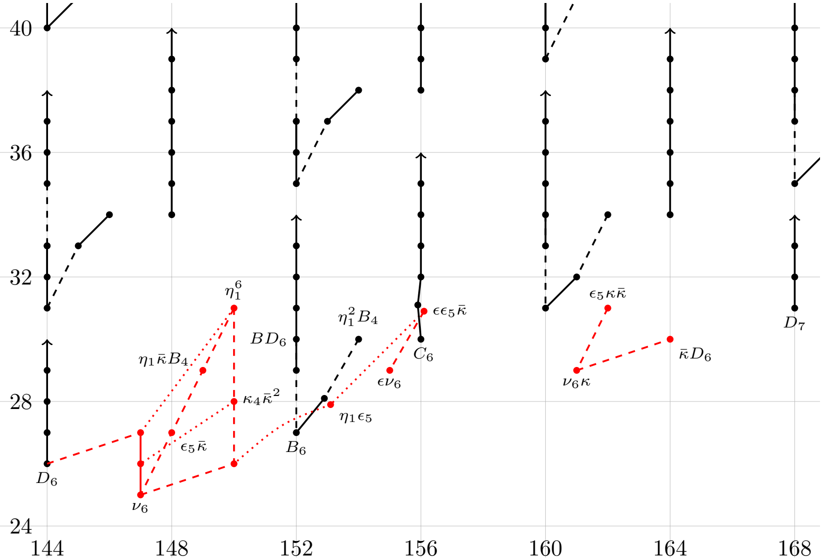
120
124
128
132
136
140
144
D_5
$\eta_1\eta_4$
ν_5
$\kappa\kappa_4$
η_1^5
BD_5
B_5
ϵ_5
$\kappa_4\bar{\kappa}$
$\epsilon\nu_5$
C_5
$\eta_1\kappa_4$
$\epsilon\epsilon_5$
$\nu_5\kappa$
$\kappa^2\kappa_4$
$\epsilon_5\kappa$
D_6

144 to 168


40
36
32
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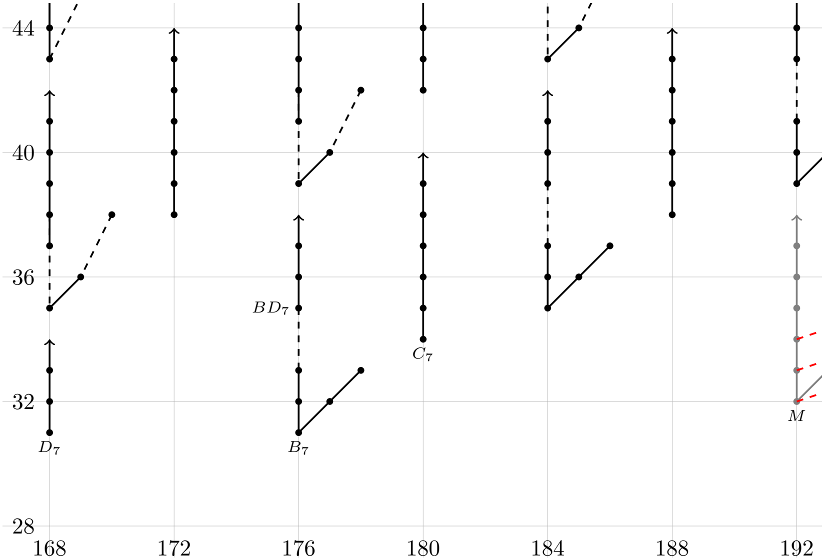
24
144
148
152
156
160
164
168
D_6
ν_6
$\eta_1 \bar\kappa B_4$
η_1^6
$\epsilon_5 \bar\kappa$
$\kappa_4 \bar\kappa^2$
BD_6
B_6
$\eta_1^2 B_4$
$\eta_1 \epsilon_5$
$\epsilon\nu_6$
C_6
$\epsilon\epsilon_5\bar\kappa$
$\epsilon_5 \kappa\bar\kappa$
$\nu_6\kappa$
$\bar\kappa D_6$
D_7

168 to 192



Definition

If $x \in \pi_d R$ and M is an R-module, let

$$M[1/x] = hocolim \left(M \xrightarrow{x} \Sigma^{-d} M \xrightarrow{x} \Sigma^{-2d} M \xrightarrow{x} \cdots \right)$$

and let M/x^∞ be the homotopy cofiber of $M \longrightarrow M[1/x]$.

We can iterate this to get $M/(x^\infty, y^\infty) = M/x^\infty \wedge_R M/y^\infty$, etcetera.

We get

$$0 \longrightarrow \pi_*(M)/x^\infty \longrightarrow \pi_*(M/x^\infty) \longrightarrow \Gamma_x \pi_{*-1}(M) \longrightarrow 0$$

Theorem

$\Sigma^{20} tmf \;\simeq\; I(tmf/(2^\infty, B^\infty, M^\infty))$

Sketch proof:

- Let N_* be the $\mathbb{Z}[B]$ submodule of $\pi_*(tmf)$ generated in degrees less than 192 (equivalently, ≤ 180).
- $\Gamma_B N_*$ is zero outside $3 \leq * \leq 164$
- N_*/B^∞ is zero above dimension 172 and is $\mathbb{Z}$, generated by C_7/B, in degree 172.
- Multiplication $N_* \otimes \mathbb{Z}[M] \longrightarrow \pi_*(tmf)$ is a $\mathbb{Z}[B, M]$-isomorphism.
- $\Gamma_M \pi_*(tmf) = 0$
- $N_* \otimes \mathbb{Z}[M]/M^\infty \cong \pi_*(tmf)/M^\infty \cong \pi_*(tmf/M^\infty)$
- Short exact sequence

$$0 \longrightarrow N_*/B^\infty \otimes Z[M]/M^\infty \longrightarrow \pi_*(tmf/(B^\infty, M^\infty))$$
$$\longrightarrow \Gamma_B N_{*-1} \otimes \mathbb{Z}[M]/M^\infty \longrightarrow 0$$

Sketch proof:(cont)

- $\pi_*(tmf/(B^\infty, M^\infty))$ is concentrated in degrees ≤ -20 and in degree -20 is $\mathbb{Z}$ generated by C_7/BM.
- It is 0 in degree -21, so the short exact sequence

$$0 \longrightarrow \pi_*(tmf/(B^\infty, M^\infty))/2^\infty \longrightarrow \pi_*(tmf/(2^\infty, B^\infty, M^\infty)) \longrightarrow \Gamma_2\pi_{*-1}(tmf/(B^\infty, M^\infty)) \longrightarrow 0$$

  implies that $\pi_*(tmf/(2^\infty, B^\infty, M^\infty))$ is concentrated in degrees ≤ -20 and in degree -20 is $\mathbb{Z}/2^\infty$.
- $\pi_*(I(tmf/(2^\infty, B^\infty, M^\infty)))$ is concentrated in degrees ≥ 20 and is $\mathbb{Z}_2$ in degree 20.
- Choosing a 2-adic generator, we get a *tmf*-module map inducing an isomorphism in π_{20} between 20-connected spectra

$$\Sigma^{20} tmf \longrightarrow I(tmf/(2^\infty, B^\infty, M^\infty))$$

- It is an equivalence by inducing along $tmf \longrightarrow BP\langle 2\rangle$ □

As usual, the equivalence of spectra yields isomorphisms (and pairings) with different shifts on the homotopy. Filter $\pi_*(tmf)$ by

$$\Delta\pi_*(tmf) \subset \Gamma_B\pi_*(tmf) \subset \Gamma_2\pi_*(tmf) \subset \pi_*(tmf)$$

where

- Δ is the submodule of Γ_B generated by the classes not in degrees 3 (mod 24), and
- $ko[k]$ is the $\mathbb{Z}[B]$ submodule generated by $\{D_k, B_k, C_k\}$ together with the appropriate η's:

0: η, η^2

1: $\eta_1, \eta\eta_1$

2: $\eta B_2, \eta_1^2$

3: $\eta B_3, \eta^2 B_3$

4: $\eta_4, \eta\eta_4$

5: $\eta B_5, \eta_1\eta_4$

6: $\eta B_6, \eta^2 B_6$

7: $\eta B_7, \eta^2 B_7$

Proposition

As a $\mathbb{Z}[B, M]$ *module*

$$\frac{\Gamma_B \pi_*(tmf)}{\Delta \pi_*(tmf)} \cong \bigoplus_{k=0}^{7} \langle \nu_k \rangle \otimes \mathbb{Z}[M]$$

and

$$\frac{\pi_*(tmf)}{\Gamma_B \pi_*(tmf)} \cong \bigoplus_{k=0}^{7} ko[k] \otimes \mathbb{Z}[M]$$

Duality in the B-torsion

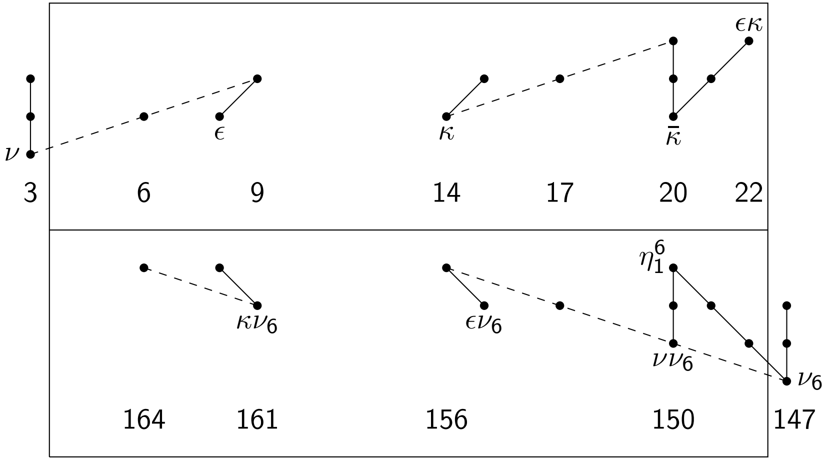


Figure: Duality between $\Delta[0]$ and $\Delta[6]$

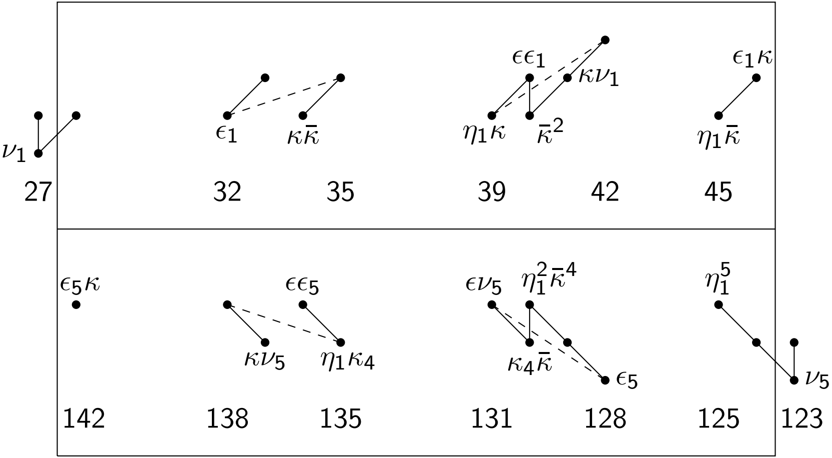


Figure: Duality between $\Delta[1]$ and $\Delta[5]$

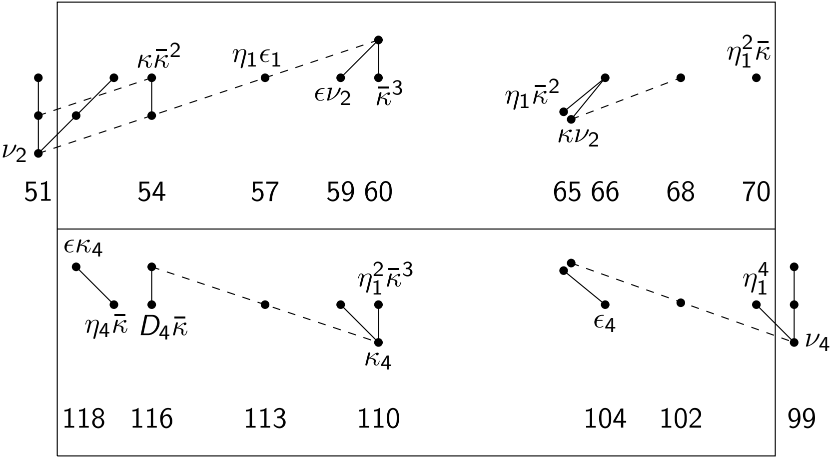


Figure: Duality between $\Delta[2]$ and $\Delta[4]$

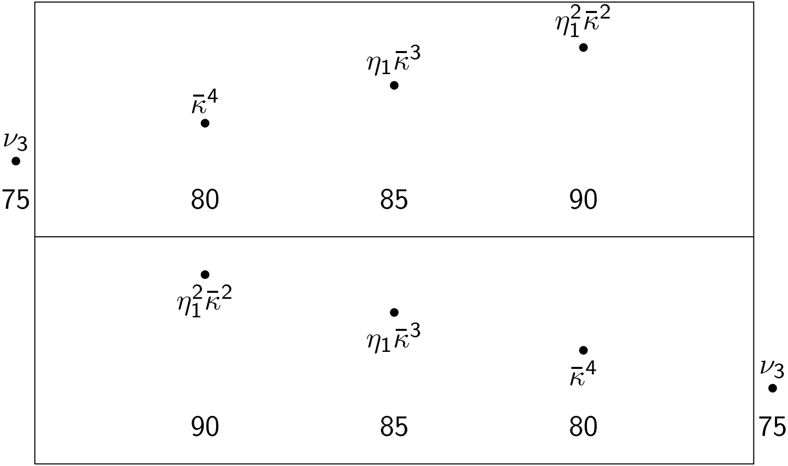


Figure: Self-duality of $\Delta[3]$

Proposition

$\langle \nu_{7-k} \rangle$ is Pontrjagin 171-dual to $\langle B_k/B \rangle$:

$$\bigoplus_{k=0}^{7} \langle \nu_k \rangle_{171-*} \cong \mathrm{Hom}\left(\bigoplus_{k=0}^{7} \langle B_k/B \rangle, \mathbb{Q}/\mathbb{Z} \right)$$

(Note however that $\nu_7 = 0$ and $0 = \langle B_0/B \rangle \subset ko[0]/B^\infty$.)
Further, in $\pi_*(tmf/B^\infty)$ the class which maps to $\nu_k \in \Gamma_B \pi_*(tmf)$ lifts to a class $\widetilde{\nu}_k$ with $2^j \widetilde{\nu}_k = C_k/B$.

Maps to $MF_{*/2}$

- The elliptic spectral sequence of Hopkins (2002) has edge hom

$$e\colon \pi_*(tmf) \longrightarrow MF_{*/2} = \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 = 1728\Delta)$$

- $MF_{*/2}$ is the ring of integral modular forms, with c_4, c_6 and Δ in weights $*/2 = 4$, 6 and 12, corresponding to topological degrees $* = 8$, 12 and 24.
- By Hopkins (2002) and Bauer (2008), $\mathrm{im}(e)$ is additively $\mathbb{Z}\{a_{i,j,k} c_4^i c_6^j \Delta^k \mid i \geq 0, j \in \{0,1\}, k \geq 0\}$ where

$$a_{i,j,k} = \begin{cases} 24/\gcd(k,24) & \text{for } i = 0 \text{ and } j = 0, \\ 1 & \text{for } i \geq 1 \text{ and } j = 0, \\ 2 & \text{for } j = 1. \end{cases}$$

This is an integral result. See also Douglas-Francis-Henriques-Hill (2014) and Konter (2012).

Proposition

- $\ker(e) = \Gamma_2 \pi_*(tmf)$.
- B_k *can be chosen to map to* $c_4 \Delta^k$, *for each* $0 \leq k \leq 7$.
- C_k *can be chosen to map to* $2c_6 \Delta^k$, *for each* $0 \leq k \leq 7$.
- D_k *can be chosen to map to* $2^i \Delta^k$ *for* $1 \leq k \leq 7$, *where*

$$i = \begin{cases} 3 & k \equiv 1 \pmod 2 \\ 2 & k \equiv 2 \pmod 4 \\ 1 & k = 4 \end{cases}$$

(We complete at 2 here.)

- *The image in* $MF_{*/2}$ *and the Adams representative in* $E_\infty(tmf)$ *uniquely determines each of* B_k, C_k *and* D_k *in* $\pi_*(tmf)$, *with the exception of* C_2, B_3 *and* C_6. *In each case, the ambiguity is a class of order 2.*

- C_2 is determined modulo $2\bar{\kappa}^3 = \nu^3\nu_2 = \eta\epsilon\nu_2$,
- B_3 is determined modulo $\bar{\kappa}^4$, and
- C_6 is determined modulo $\nu^3\nu_6 = \eta\epsilon\nu_6$.

General setup

- The Davis-Mahowald spectral sequence is a substitute for the Cartan-Eilenberg spectral sequence when the sub Hopf algebra is not normal.
- In Davis and Mahowald (1982) the multiplicative structure is a somewhat *ad hoc* afterthought. We give precise conditions for it.
- Γ, Hopf algebra over k
- $\Lambda \subset \Gamma$, sub Hopf algebra
- $\Omega := \Gamma /\!\!/ \Lambda = \Gamma \otimes_\Lambda k$ is a quotient Γ-module *coalgebra*

DMSS, dual formulation

- Γ_* commutative Hopf algebra.
- Λ_* quotient Hopf algebra of Γ.
- $\Omega_* = \Gamma_* \Box_{\Lambda_*} k$ left Γ_*-comodule *algebra*.
- Require (suitable) Γ_*-comodule algebra resolution $k \to (\Omega_* \otimes R^*, d)$.
- Get multiplicative Davis–Mahowald spectral sequence

$$E_1^{\sigma,s,*} = Ext_{\Lambda_*}^{s,*}(k, R^\sigma) \Longrightarrow_\sigma Ext_{\Gamma_*}^{s+\sigma,*}(k, k)\,.$$

- Untwisting $\Omega_* \otimes R^\sigma \cong \Gamma_* \Box_{\Lambda_*} R^\sigma$ is multiplicative for commutative Γ_*.

DMSS, dual formulation, cont.

- Assume a graded Γ_*-comodule algebra $R^* = \bigoplus_\sigma R^\sigma$ and homomorphisms $d\colon \Omega_* \otimes R^\sigma \to \Omega_* \otimes R^{\sigma+1}$
- such that $(\Omega_* \otimes R^*, d)$ is a differential graded Γ_*-comodule algebra and the unit $k \to (\Omega_* \otimes R^*, d)$ is a quasi-isomorphism.
- Get an algebra spectral sequence

$$E_1^{\sigma,s} = Ext^s_{\Lambda_*}(k, R^\sigma) \Longrightarrow_\sigma Ext^{s+\sigma}_{\Gamma_*}(k,k)$$

- Product $E_1^{\sigma,*} \otimes E_1^{\tau,*} \to E_1^{\sigma+\tau,*}$ equals pairing induced by Λ_*-comodule product $R^\sigma \otimes R^\tau \to R^{\sigma+\tau}$.

Main example: $A(2)$

- Commutative Hopf algebras

$$\Gamma_* = A(2)_* \twoheadrightarrow A(1)_* = \Lambda_*$$

i.e.,

$$\mathbb{F}_2[\xi_1, \bar{\xi}_2, \bar{\xi}_3]/(\xi_1^8, \bar{\xi}_2^4, \bar{\xi}_3^2) \twoheadrightarrow \mathbb{F}_2[\xi_1, \bar{\xi}_2]/(\xi_1^4, \bar{\xi}_2^2)$$

- The left $A(2)_*$-comodule algebra

$$\Omega_* = A(2)_* \,\square_{A(1)_*}\, \mathbb{F}_2 = E[\xi_1^4, \bar{\xi}_2^2, \bar{\xi}_3]$$

is a sub $A(2)_*$-comodule algebra, but not a sub coalgebra.

Main example, cont.

- Resolve by $A(2)_*$-comodule algebra $R^* = \mathbb{F}_2[x_4, x_6, x_7]$ with coaction

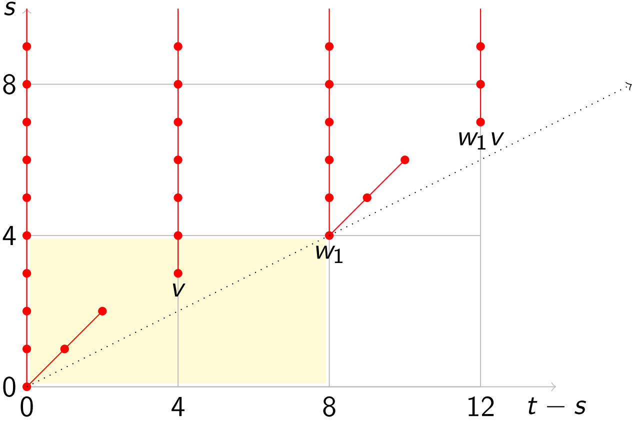
$$\begin{aligned}\nu(x_4) &= 1 \otimes x_4\\ \nu(x_6) &= 1 \otimes x_6 + \xi_1^2 \otimes x_4\\ \nu(x_7) &= 1 \otimes x_7 + \xi_1 \otimes x_6 + \bar{\xi}_2 \otimes x_4.\end{aligned}$$

- Resolution $\mathbb{F}_2 \to \Omega_* \otimes R^*$ has differential

$$\begin{aligned}d(\xi_1^4) &= x_4\\ d(\bar{\xi}_2^2) &= x_6\\ d(\bar{\xi}_3) &= x_7\end{aligned}$$

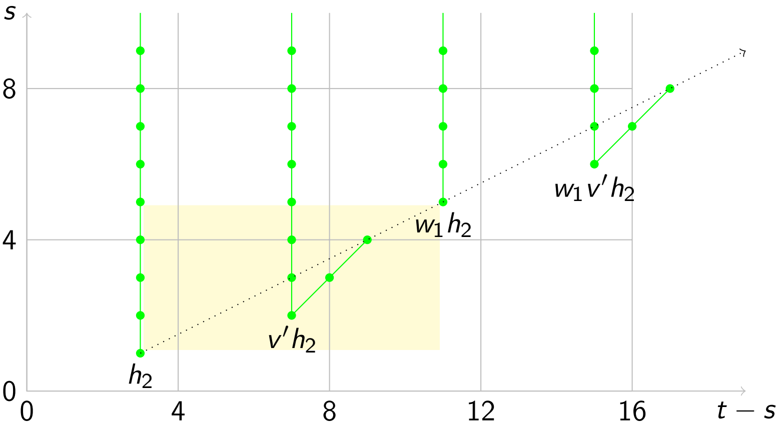
$\sigma = 0$

- $R^0 = \mathbb{F}_2$
- $E_1^{0,*,*} = Ext^{*,*}_{A(1)_*}(\mathbb{F}_2, \mathbb{F}_2) = ko^{*,*}$.



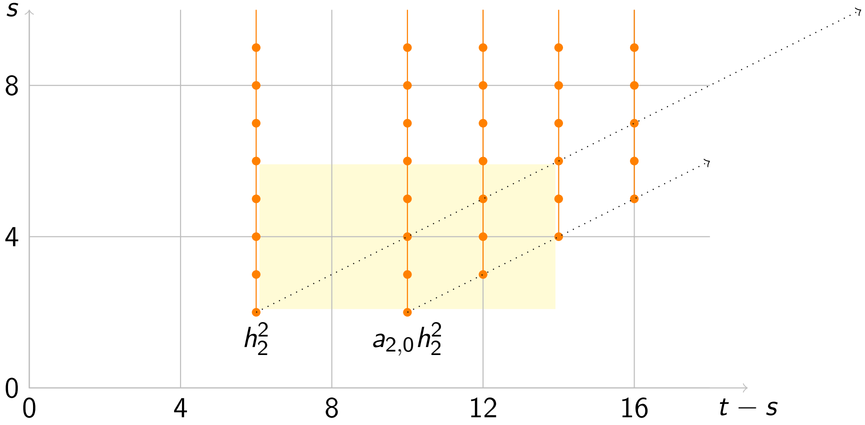
$\sigma = 1$

- $R^1 = \mathbb{F}_2\{x_4, x_6, x_7\} = \Sigma^4 H_*(S \cup_\eta e^2 \cup_2 e^3)$.
- $E_1^{1,*,*} = Ext_{A(1)_*}^{*,*}(\mathbb{F}_2, R^1) = ksp^{*,*}\{h_2\}$.



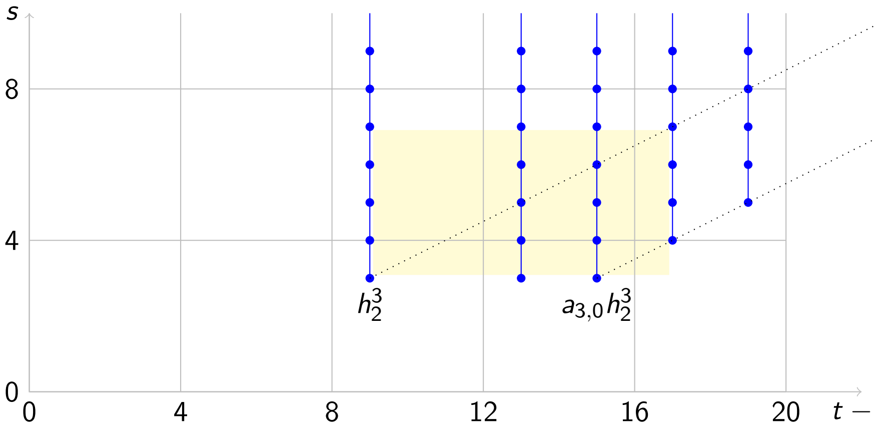
$\sigma = 2$

- $R^2 = \mathbb{F}_2\{x_4^2, x_4x_6, x_4x_7, x_6^2, x_6x_7, x_7^2\}$.
- $E_1^{2,*,*} = Ext^{*,*}_{A(1)_*}(\mathbb{F}_2, R^2) = G_2^{*,*}\{h_2^2\}$.



$\sigma = 3$

- $\dim R^3 = 10$.
- $E_1^{3,*,*} = Ext_{A(1)_*}^{*,*}(\mathbb{F}_2, R^3) = G_3^{*,*}\{h_2^3\}$.



$\mathrm{Ext}_{\mathcal{A}(2)}$

Theorem (Shimada and Iwai)

The cohomology of $A(2)$ *is*

$$\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, h_1, h_2, c_0, d_0, e_0, g, \alpha, \beta, \gamma, \delta, w_1, w_2]/I.$$

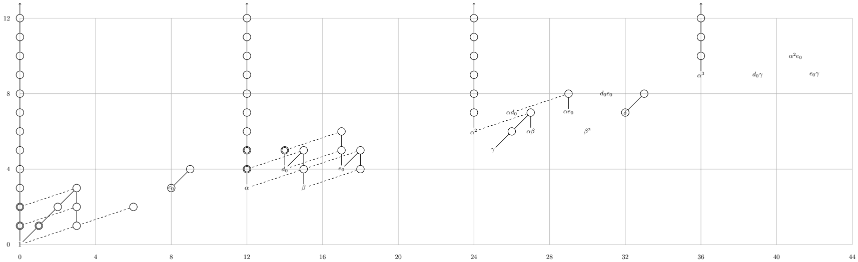
The ideal I has 54 generators:

- $h_0 h_1,\ h_1 h_2,\ h_0^2 h_2 - h_1^3,\ h_0 h_2^2,\ h_2^3$
- ...
- $c_0\gamma - h_1\delta,\ \beta\gamma - g^2,\ d_0^2 - gw_1,\ \gamma\delta - h_1 c_0 w_2,$
- $\gamma^2 - h_1^2 w_2 - g\beta^2,\ \alpha^4 - h_0^4 w_2 - w_1 g^2$

- Free over $\mathbb{F}_2[w_1, w_2]$; here w_1 and w_2 restrict to v_1^4 and v_2^8, resp.
- A sum of cyclic $R = \mathbb{F}_2[g, w_1, w_2]$-modules isomorphic to R, $R/(g)$ and $R/(g^2)$.
- Four infinite families, $h_0^i \alpha^j$, $i \geq 0$, $0 \leq j \leq 3$.
- Thirty-two other summands.
- E_3, E_4 and $E_5 = E_\infty$ are then modules over $R_1 = \mathbb{F}_2[g, w_1, w_2^2]$ and $R_2 = \mathbb{F}_2[g, w_1, w_2^4]$ resp. Mostly cyclic.

R_0 generators of $\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$

No circle indicates an R_0, one circle an $R_0/(g)$, and two circles an $R_0/(g^2)$.



First differentials

Squaring operations in Ext quickly give us quite a few differentials.

- $d_2(\alpha) = h_2 w_1$ and $d_2(\beta) = h_0 d_0$
- $d_3(\alpha^2) = h_1 d_0 w_1$ and $d_3(\beta^2) = h_1 g w_1$
- $d_3(w_2^2) = Sq^9(d_2(w_2))$

From these many others follow by the Leibniz rule.

Key differentials

There are three *hard* differentials, from which we can deduce everything else using the product structure. They are

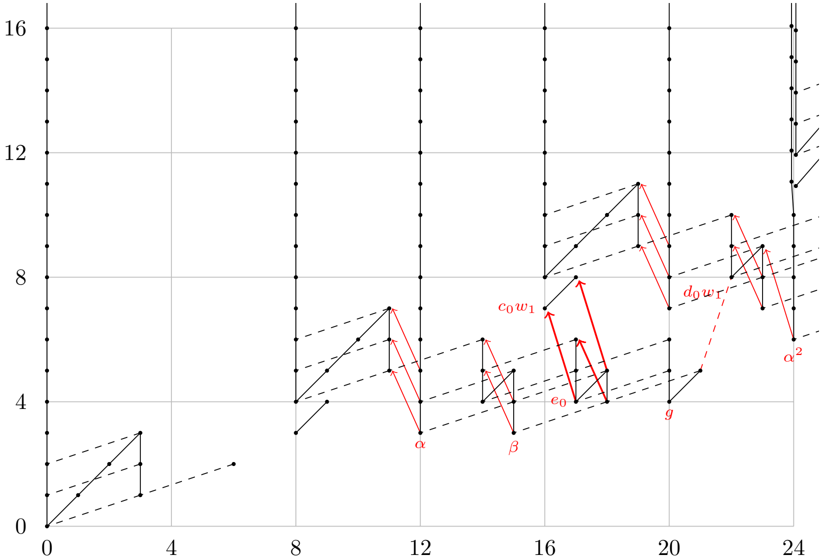
Theorem

- $d_3(e_0) = c_0 w_1$
- $d_4(e_0 g) = g w_1^2$
- $d_2(w_2) = \alpha\beta g$

$d_3(e_0) = c_0 w_1$

- The $\mathrm{Im(J)}$ generator $\rho \in \pi_{15}(S)$ in Adams filtration 4 must either map to 0 or $\eta\kappa$ in $\pi_{15}(tmf)$.
- $\eta\rho \in \pi_*(S)$ is detected by $\{Pc_0\}$ in $\pi_*(S)$, which maps to $c_0 w_1$ in $\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$.
- $\eta^2\kappa = 0$ in $\pi_*(S)$ (Toda).
- $c_0 w_1$ must be a boundary and $d_3(e_0)$ is the only chance.

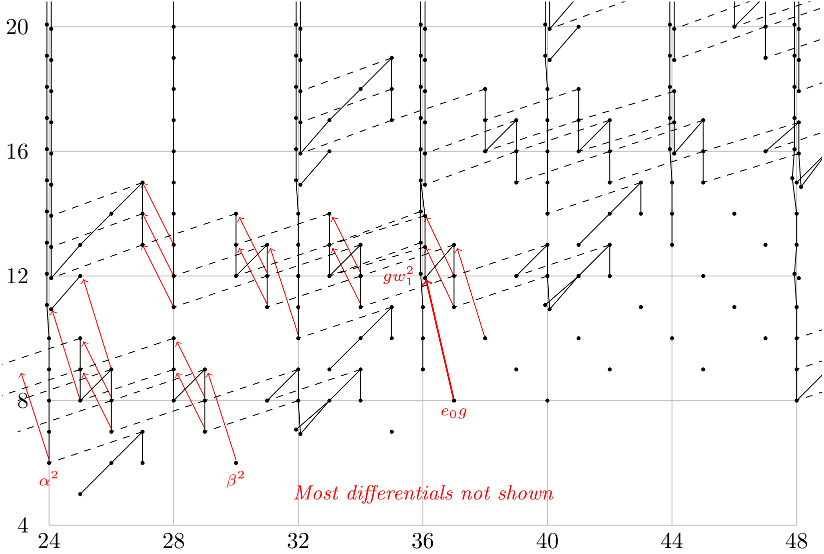
0 to 24



$d_4(e_0 g) = g w_1^2$

- $\eta^2\bar{\kappa}$ is detected by Pd_0 in $\pi_{22}(S)$ (Barratt-Mahowald-Tangora, Mimura?). This maps to $d_0 w_1$ in $\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$.
- $\kappa \cdot \eta^2\bar{\kappa} = 0$ since $\eta^2\kappa = 0$
- $\kappa \cdot \eta^2\bar{\kappa}$ is detected by $d_0 \cdot Pd_0$ which maps to $d_0^2 w_1 = g w_1^2$ in $\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$.
- $d_4(e_0 g)$ is the only class which can hit it.

24 to 48


20
16
12
8
4
24
28
32
36
40
44
48
α^2
β^2
gw_1^2
$e_0 g$
Most differentials not shown

$d_2(w_2) = \alpha\beta g$

Corollary

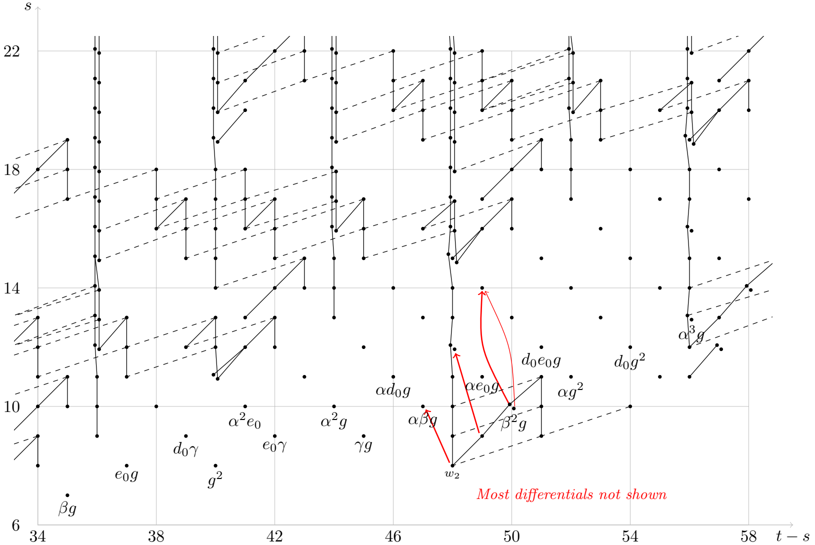
$d_4(d_0 e_0) = d_0 w_1^2$ and $d_4(\beta^2 g) = \alpha^2 e_0 w_1$ and these are both nonzero.

Theorem

$d_4(h_1^2 w_2) = \alpha^2 e_0 w_1$, $d_2(w_2) = \alpha\beta g$, and $d_3(h_1 w_2) = g^2 w_1$.

- γ must live to at least E_6, so $d_4(\gamma^2) = 0$
- $\gamma^2 = h_1^2 w_2 + \beta^2 g$, so $d_4(h_1^2 w_2) = \alpha^2 e_0 w_1 \neq 0$
- If $d_2(w_2) = 0$ then $d_4(h_1^2 w_2) = 0$, contradiction, and $d_2(w_2) = \alpha\beta g$ is the only possibility.
- If $d_3(h_1 w_2) = 0$ then $d_4(h_1^2 w_2) = 0$, contradiction, and $d_3(h_1 w_2) = g^2 w_1$ is the only possibility.

34 to 58


s
22
18
14
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38
42
46
50
54
58
t − s
βg
e_0g
$d_0\gamma$
g^2
α^2e_0
$e_0\gamma$
α^2g
γg
αd_0g
αβg
w_2
αe_0g
β^2g
d_0e_0g
αg^2
d_0g^2
α^3g
Most differentials not shown

Thank you