DEFINITIONS

Degree A degree is the $\frac{1}{180}$th part of a straight angle.

Right Angle A $90°$ angle is called a right angle.

Perpendicular Two lines are called perpendicular if they form a right angle.

Congruent Triangles Two triangles $\triangle ABC$ and $\triangle DEF$ are congruent (written $\triangle ABC \cong \triangle DEF$) if all three corresponding angles and all three corresponding sides are equal.

Similar Triangles Two triangles $\triangle ABC$ and $\triangle DEF$ are similar (written $\triangle ABC \sim \triangle DEF$) if all three corresponding angles are equal.

Parallel Lines Two lines are parallel if they do not intersect.

Midpoint of a line segment The midpoint of a segment AB is the point M on the segment for which $MA = MB$.

Angle bisector The bisector of an angle is the line that goes through the vertex of the angle and splits the angle into two equal parts.

Parallelogram A quadrilateral is a parallelogram if the opposite sides are parallel.

Rectangle A quadrilateral is a rectangle if it has four right angles.

Square A quadrilateral is a square if it has four equal sides and four right angles.

Distance from a point to a line The distance from a point P to a line m is defined to be the length of the line segment from P to m which is *perpendicular* to m.

Base and height of a triangle Any side can be chosen as the base. Once we have chosen the base, the height is the distance from the remaining vertex to the line containing the base.

Definition of concurrent lines Three lines are *concurrent* if they meet at a single point.

Definition of perpendicular bisector The perpendicular bisector of a line segment is the line that goes through the midpoint and is perpendicular to the segment.

Definition of circumcenter The point where the three perpendicular bisectors of the sides of a triangle meet is called the circumcenter of the triangle.

Definition of circle A circle consists of all of the points which are at a given distance (called the *radius*) from a given point (called the *center*).

Definition of incenter The point where the three angle bisectors meet is called the incenter of the triangle.

Definition of altitude An altitude of a triangle is a line that goes through a vertex of the triangle and is perpendicular to the opposite side.

Definition of orthocenter The point where the three altitudes meet is called the orthocenter of the triangle.

Definition of median A median of a triangle is a line that goes through a vertex of the triangle and through the midpoint of the opposite side.

Definition of centroid The point where the three medians meet is called the centroid of the triangle.

Definition of collinear Three points are said to be collinear if they all lie on the same line.

Definition of signed ratio Let A, B and C be three points on a line. The symbol $\dfrac{\overset{\longmapsto}{CA}}{\overset{\longmapsto}{CB}}$ is equal to $\dfrac{CA}{CB}$ if C is outside of the segment AB and is equal to $-\dfrac{CA}{CB}$ if C is inside the segment AB.

Degree of arc-measure A degree of arc on a circle is the $\frac{1}{360}$th part of the full circle.

Definition of tangent line A line is tangent to a circle $\Longleftrightarrow$ it intersects the circle in exactly one point.

BASIC FACTS

BF 1 SSS: if two triangles have three pairs of corresponding sides equal, then the triangles are congruent.

BF 2 SAS: if two triangles have two pairs of corresponding sides and the included angles equal, then the triangles are congruent.

BF 3 ASA: if two triangles have two pairs of corresponding angles and the included side equal, then the triangles are congruent.

BF 4 If two triangles are similar then their corresponding sides are proportional: that is, if $\triangle ABC$ is similar to $\triangle DEF$ then

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

BF 5 If two parallel lines ℓ and m are crossed by a transversal, then all corresponding angles are equal. If two lines ℓ and m are crossed by a transversal, and at least one pair of corresponding angles are equal, then the lines are parallel.

BF 6 The whole is the sum of its parts; this applies to lengths, angles, areas and arcs.

BF 7 Through two given points there is one and only one line. (This means two things. First, it is possible to draw a line through two points. Second, if two lines have two or more points in common they must really be the same line).

BF 8 On a ray there is exactly one point at a given distance from the endpoint. (This means two things. First, it is possible to find a point on the ray at a given distance from the endpoint. Second, if two points on the ray have the same distance from the endpoint they must really be the same point.)

BF 9 It is possible to extend a line segment to an infinite line.

BF 10 It is possible to find the midpoint of a line segment.

BF 11 It is possible to draw the bisector of an angle.

BF 12 Given a line ℓ and a point P (which may be either on ℓ or not on ℓ) it is possible to draw a line through P which is perpendicular to ℓ.

BF 13 Given a line ℓ and a point P not on ℓ, it is possible to draw a line through P which is parallel to ℓ.

BF 14 If two lines are each parallel to a third line then they are parallel to each other.

BF 15 The area of a rectangle is the base times the height.

THEOREMS

Theorem 1. *When two lines cross,*

(a) *adjacent angles add up to* $180°$*, and*

(b) *vertical angles are equal.*

Theorem 2. *Suppose that ℓ and m are two lines crossed by a transversal.*

(a) *If ℓ and m are parallel, then both pairs of alternate interior angles are equal. If at least one pair of alternate interior angles are equal, then ℓ and m are parallel.*

(b) *If ℓ and m are parallel, then each pair of interior angles on the same side of the transversal adds up to* $180°$*. If at least one pair of interior angles on the same side of the transversal adds up to* $180°$*, then ℓ and m are parallel.*

(c) *If ℓ and m are parallel, then each pair of exterior angles on the same side of the transversal adds up to 180°. If at least one pair of exterior angles on the same side of the transversal adds up to 180°, then ℓ and m are parallel.*

Theorem 3. *The angles of a triangle add up to 180°.*

Theorem 4. *If two triangles ABC and DEF have $\angle A = \angle D$ and $\angle B = \angle E$ then also $\angle C = \angle F$.*

Theorem 5. (a) *If two sides of a triangle are equal then the opposite angles are equal.*

(b) *If two angles of a triangle are equal then the opposite sides are equal.*

Theorem 6. *In triangle ABC, if $\angle B$ is a right angle then the area of the triangle is $\frac{1}{2}AB \cdot BC$.*

Theorem 7. *The area of a triangle is one-half of the base times the height.*

Theorem 8. *If $\triangle ABC \sim \triangle DEF$ and $\dfrac{AC}{DF} = r$ then the area of $\triangle ABC$ is r^2 times the area of $\triangle DEF$.*

Theorem 9 (Pythagorean theorem)**.** *In a right triangle the sum of the squares of the two legs is equal to the square of the hypotenuse.*

Theorem 10 (Hypotenuse-Leg Theorem)**.** *In triangles ABC and DEF, if $\angle A$ and $\angle D$ are right angles, and if $BC = EF$ and $AB = DE$, then $\triangle ABC \cong \triangle DEF$. That is, if two right triangles have the hypotenuse and a leg matching then they are congruent.*

Theorem 11. *If $ABCD$ is a parallelogram then opposite sides of $ABCD$ are equal.*

Theorem 12. *If $ABCD$ is a parallelogram then opposite angles of $ABCD$ are equal.*

Theorem 13. *If a quadrilateral has a pair of sides which are equal and parallel then it is a parallelogram.*

Theorem 14. *A quadrilateral is a parallelogram $\iff$ the diagonals bisect each other (that is, $\iff$ the intersection of the two diagonals is the midpoint of each diagonal).*

Theorem 15. *Suppose that ℓ and m are parallel lines. If A and B are any points on ℓ, and if C and D are points on m with $AC \perp m$ and $BD \perp m$ then $AC = BD$.*

Theorem 16. (a) *Suppose that C is a point on the segment AB. C is the midpoint of $AB \iff AB = 2AC$.*

(b) *A line segment can have only one midpoint.*

Theorem 17. *In triangle ABC, let D be the midpoint of AC and suppose that E is a point on BC with DE parallel to AB. Then E is the midpoint of BC and $DE = \frac{1}{2}AB$.*

Theorem 18. *In triangle ABC, let D be the midpoint of AC and let E be the midpoint of BC. Then DE is parallel to AB and $DE = \frac{1}{2}AB$.*

Theorem 19 (SAS for similarity)**.** *In triangles ABC and DEF, if $\angle C = \angle F$ and $\dfrac{AC}{DF} = \dfrac{BC}{EF}$ then $\triangle ABC \sim \triangle DEF$.*

Theorem 20 (SSS for similarity)**.** *In triangles ABC and DEF, if $\dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$ then $\triangle ABC \sim \triangle DEF$.*

Theorem 21. $\sin(\angle ABC) = \sin(180° - \angle ABC)$

Theorem 22. *For any triangle ABC, the area can be calculated by any of the following three formulas:*

$$\text{area of } \triangle ABC = \frac{1}{2} AB \cdot AC \cdot \sin \angle A$$
$$\text{area of } \triangle ABC = \frac{1}{2} AB \cdot BC \cdot \sin \angle B$$
$$\text{area of } \triangle ABC = \frac{1}{2} AC \cdot BC \cdot \sin \angle C$$

that is, the area is one half the product of two sides times the sine of the included angle.

Theorem 23 (Law of Sines)**.** *In any triangle ABC,*

$$\frac{\sin(\angle A)}{BC} = \frac{\sin(\angle B)}{AC} = \frac{\sin(\angle C)}{AB}$$

Theorem 24. *For any triangle ABC, the perpendicular bisectors of AB, AC and BC are concurrent.*

Theorem 25. *Given a triangle ABC with circumcenter O, suppose that $\mathcal{C}$ is a circle with center O that goes through one of the vertices of the triangle. Then C also goes through the other two vertices.*

Theorem 26. *For any triangle ABC, the bisectors of $\angle A$, $\angle B$ and $\angle C$ are concurrent.*

Theorem 27. *For any triangle ABC, the three altitudes are concurrent.*

Theorem 28. *The point where two medians of a triangle intersect is 2/3 of the way from each of the two vertices to the opposite midpoint.*

Theorem 29. *For any triangle ABC, the three medians are concurrent.*

Theorem 30. *Let ABC be any triangle. Let O be the circumcenter of ABC, let G be the centroid of ABC, and let H be the orthocenter of ABC. Then O, G and H are collinear.*

Theorem 31 (Theorem of Menelaus)**.** *Let ABC be any triangle. Let A' be a point of $\overleftrightarrow{BC}$ other than B and C, let B' be a point of $\overleftrightarrow{AC}$ other than A and C, and let C' be a point of $\overleftrightarrow{AB}$ other than A and B. If A', B' and C' are collinear then*

$$\frac{A'B}{A'C}\,\frac{B'C}{B'A}\,\frac{C'A}{C'B} = 1$$

Theorem 32. *Let A, B, and C be three points on a number line, and let a, b and c be their coordinates. Then*

$$\frac{\overset{\longmapsto}{CA}}{\overset{\longmapsto}{CB}} = \frac{c-a}{c-b}.$$

Theorem 33. *Let ℓ be any line, and let A, B, C' and C'' be points on ℓ, with A not the same point as B and with C' and C'' different from both A and B. If $\dfrac{\overset{\longmapsto}{C'A}}{\overset{\longmapsto}{C'B}} = \dfrac{\overset{\longmapsto}{C''A}}{\overset{\longmapsto}{C''B}}$ then C' is the same point as C''.*

Theorem 34. *Let ABC be a triangle. Let A' be a point of $\overleftrightarrow{BC}$ other than B and C, let B' be a point of $\overleftrightarrow{AC}$ other than A and C, and let C' be a point of $\overleftrightarrow{AB}$ other than A and B. If A', B' and C' are collinear then*

$$\frac{\overset{\longmapsto}{A'B}}{\overset{\longmapsto}{A'C}}\,\frac{\overset{\longmapsto}{B'C}}{\overset{\longmapsto}{B'A}}\,\frac{\overset{\longmapsto}{C'A}}{\overset{\longmapsto}{C'B}} = 1.$$

Theorem 35. *Let ABC be a triangle. Let A' be a point of $\overleftrightarrow{BC}$ other than B and C, let B' be a point of $\overleftrightarrow{AC}$ other than A and C, and let C' be a point of $\overleftrightarrow{AB}$ other than A and B. If*

$$\frac{\overset{\longmapsto}{A'B}}{\overset{\longmapsto}{A'C}}\,\frac{\overset{\longmapsto}{B'C}}{\overset{\longmapsto}{B'A}}\,\frac{\overset{\longmapsto}{C'A}}{\overset{\longmapsto}{C'B}} = 1$$

then A', B' and C' are collinear.

Theorem 36 (Theorem of Ceva)**.** *Let ABC be a triangle. Let A' be a point of $\overleftrightarrow{BC}$ other than B and C, let B' be a point of $\overleftrightarrow{AC}$ other than A and C, and let C' be a point of $\overleftrightarrow{AB}$ other than A and B. If $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are concurrent then*

$$\frac{\overset{\longmapsto}{A'B}}{\overset{\longmapsto}{A'C}}\,\frac{\overset{\longmapsto}{B'C}}{\overset{\longmapsto}{B'A}}\,\frac{\overset{\longmapsto}{C'A}}{\overset{\longmapsto}{C'B}} = -1$$

Theorem 37. *Let ABC be a triangle. Let A' be a point of $\overleftrightarrow{BC}$ other than B and C, let B' be a point of $\overleftrightarrow{AC}$ other than A and C, and let C' be a point of $\overleftrightarrow{AB}$ other than A and B. If*

$$\frac{\overset{\longmapsto}{A'B}}{\overset{\longmapsto}{A'C}}\,\frac{\overset{\longmapsto}{B'C}}{\overset{\longmapsto}{B'A}}\,\frac{\overset{\longmapsto}{C'A}}{\overset{\longmapsto}{C'B}} = -1$$

then $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are concurrent.

Theorem 38. *Let A, B and C be points on a circle with center O.*

(a) *If B is outside of $\angle AOC$, then $\angle ABC = \frac{1}{2}\angle AOC$.*

(b) *If B is inside of $\angle AOC$, then $\angle ABC = 180^\circ - \frac{1}{2}\angle AOC$.*

Theorem 39. *The arc cut off by a central angle is the same number of degrees as the angle.*

Theorem 40. *Let A, B and C be points on a circle, and let ADC be the arc cut off by $\angle ABC$. Then*

$$\angle ABC = \frac{1}{2}\text{arc } ADC$$

that is, the number of degrees in angle ABC is half the number of degrees in arc ADC.

Theorem 41. *Let A, B, C and D be points on a circle and consider the angles ABC and ADC.*

(a) *If $\angle ABC$ and $\angle ADC$ cut off the same arc, then $\angle ABC = \angle ADC$.*

(b) *If $\angle ABC$ and $\angle ADC$ do not cut off the same arc, then $\angle ABC = 180^\circ - \angle ADC$.*

Theorem 42. *Let A, B, C and D be points on a circle, and suppose that the lines AB and CD meet at a point P. Then $PA \cdot PB = PC \cdot PD$.*

Theorem 43. *Let $\mathcal{C}$ be a circle with center O, let A be a point on the circle, and let m be a line through A. Then m is tangent to the circle $\Longleftrightarrow$ m is perpendicular to OA.*

Theorem 44. *Given a line ℓ and a point P not on it, it is possible to construct a circle with center at P that is tangent to ℓ.*

Theorem 45. *Given a triangle ABC with incenter I, the circle with center at I that is tangent to one of the sides is also tangent to the other two sides.*

Theorem 46. *Let A and B be points on a circle, let C be a point on the tangent line at B, and let ADB be the arc cut off by $\angle ABC$. Then*

$$\angle ABC = \frac{1}{2}\text{arc } ADB$$

Theorem 47. *If the opposite pairs of angles in a quadrilateral add to 180° then there is a circle going through all four vertices.*