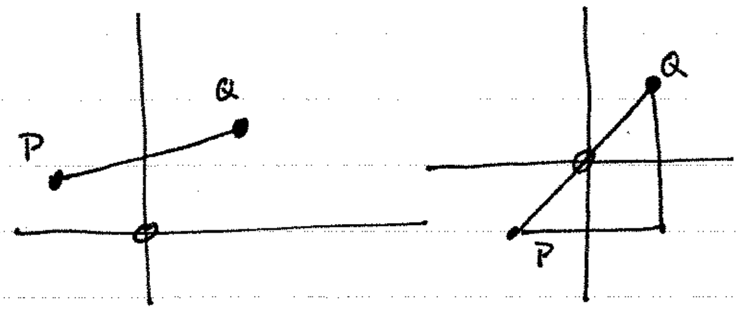
Math 5520 Winter 2010 Midterm

1. $\mathbb{R}^2 - 0$ is connected

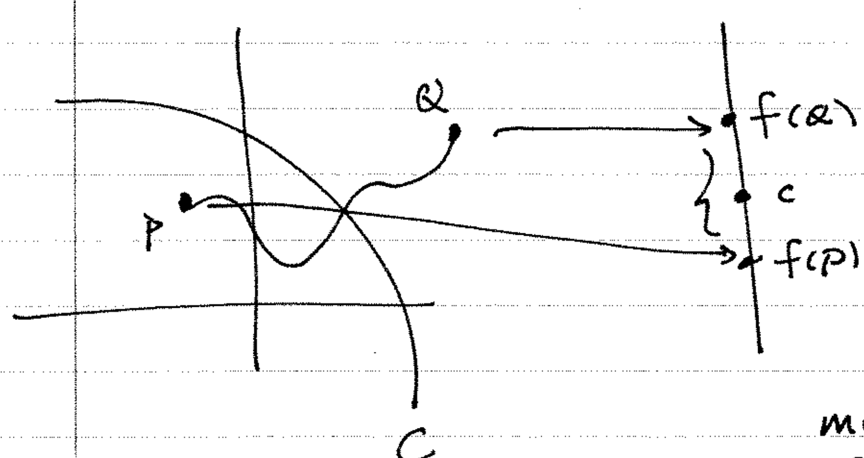
Proof: $\mathbb{R}^2 - 0$ is path connected: the straight line path $tP + (1-t)Q$ works in most cases. If it happens to



pass through 0, there is a path consisting of a horizontal line followed by a vertical line. Since path connected $\Rightarrow$ connected, we are done. //

2. (a) Suppose $P \leftarrow C$. Then $f(P) \leftarrow f(C)$. But $f(C) = \{c\}$ and the only point near $\{c\}$ is c itself. Hence $f(P) = c$ and therefore $P \in C$.

(b) Suppose $f(P) < c < f(Q)$. Let $\gamma(t)$ be any path in $\mathbb{R}^2$ from P to Q. Since paths are connected, $\{f(\gamma(t))\}$ is connected. It contains $f(P)$ and $f(Q)$, so also contains c. Hence $\gamma(t)$ must intersect C. Thus, no path from P to Q lies in $\mathbb{R}^2 - C$. //



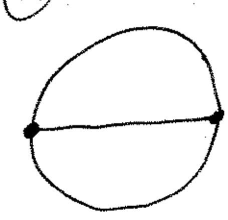
3. Any continuous $f: \mathbb{R} \to \mathbb{R}$ whose values are all integers must be constant.

Proof: If f is not constant, let $f(x) < f(y)$. Then f must take on every value between $f(x)$ and $f(y)$, and these are not all integers, contradicting our assumption on f. Hence f must be constant. //

4. If $V: S^1 \longrightarrow \mathbb{R}$ is a continuous vector field which is never 0 on the circle S^1, then $\exists\, \varepsilon > 0$ s.t. $|V(P)| > \varepsilon$ for all $P \in S^1$.

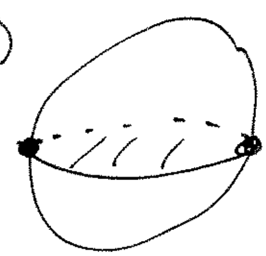
Pf: If no such ε existed, we could find points $P_n \in S^1$ such that $|V(P_n)| < 1/n$. Since S^1 is compact, there must be a point $P \leftarrow (P_1, P_2, P_3, \ldots)$. Then $|V(P)| \leftarrow (1, 1/2, 1/3, \ldots)$ and so $|V(P)| = 0$ and hence $V(P) = 0$. No such P exists, so there must have been an ε as required. //

5. (a)



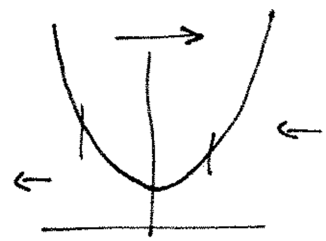
$V - E + F = 2 - 3 + 0$
$= -1$

(b)



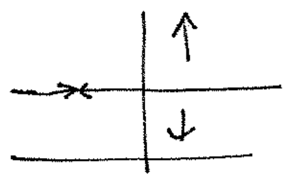
$V - E + F$
$= 2 - 2 + 3 = 3$

6. $V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y - x^2 - 1 \\ y - 2 \end{pmatrix}$

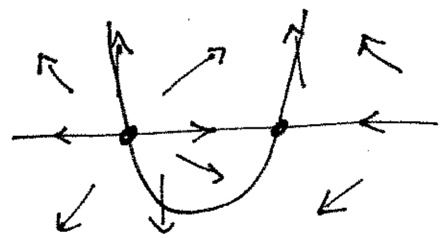


(a) Nullclines: $x' = 0 \iff y = x^2 + 1$

$y' = 0 \iff y = 2$



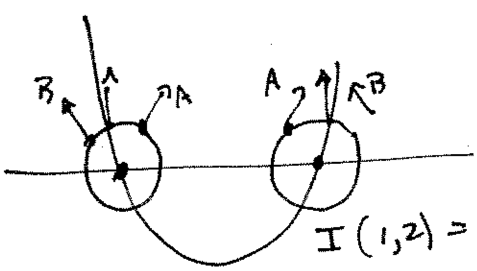
Direction of flow:



(b) Critical points $y = x^2 + 1 = 2 \implies x^2 = 1 \implies x = \pm 1$

$(1, 2)$ and $(-1, 2)$

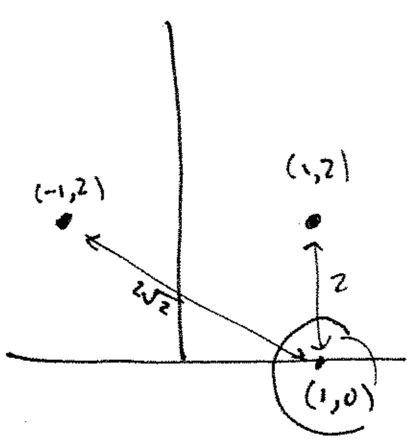
(c)



$I(1,2) = -1$

$I(-1,2) = +1$

(d)



$$W_\gamma(v) = \begin{cases} 0 & r < 2 \\ -1 & 2 < r < 2\sqrt{2} \\ 0 & r > 2\sqrt{2} \end{cases}$$

7. If $f: D^2 \to S^1$ is continuous then there is a point $P \in S^1$ such that $f(P) = P$.

Proof: The composite $D^2 \xrightarrow{f} S^1 \subset D^2$ is continuous so has a fixed point by the Brouwer fixed point theorem: $\exists\, P \in D^2$ such that $P = f(P)$. But $f(P) \in S^1$ so $P \in S^1$. //

8. If $\gamma: [0,1] \to \mathbb{R}^2$ is a continuous curve then the image of γ is contained in the disk of radius r about $(0,0)$, for some r.

Proof: $[0,1]$ is compact, so $\gamma[0,1]$ is compact. By the Bolzano-Weierstrass theorem, $\gamma[0,1]$ is bounded, i.e. contained in such a disk. //