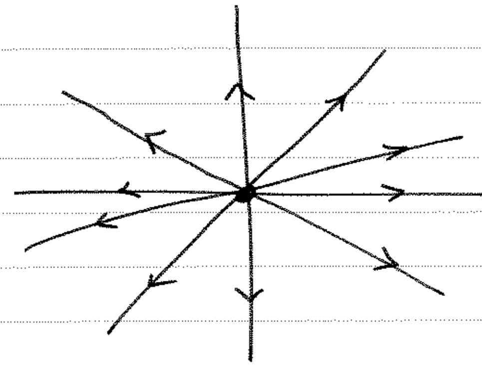
HW 6 §7 # 1 a,c,e, 5

§8 # 1,2

(1) (a) $V\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$



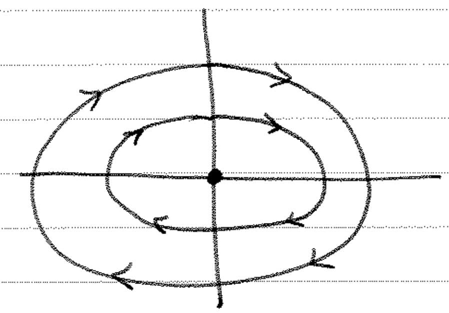
$x' = x \Rightarrow x = x_0 e^t$

$y' = y \Rightarrow y = y_0 e^t$

Then $\dfrac{y}{x} = \dfrac{y_0}{x_0}$ is constant.

$\begin{pmatrix}0\\0\end{pmatrix}$ is an unstable node

(c) $V\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2y\\-x\end{pmatrix}$



$x'' = (2y)' = -2x$

So

$P(x) = -\int -2x\,dx = x^2$

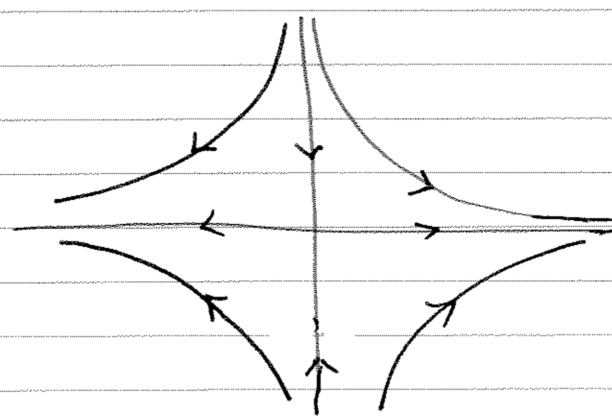
is our "potential energy".

$\begin{pmatrix}0\\0\end{pmatrix}$ is a center (stable)

Then $\frac{1}{2}(x')^2 + x^2 = \frac{1}{2}(2y)^2 + x^2 = 2y^2 + x^2$

is constant.

(e) $V\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\-y\end{pmatrix}$



$x' = x \Rightarrow x = x_0 e^t$

$y' = -y \Rightarrow y = y_0 e^{-t}$

So $xy = x_0 y_0$ is constant

saddle (unstable)

⑤ For Figure 7.4 in the book

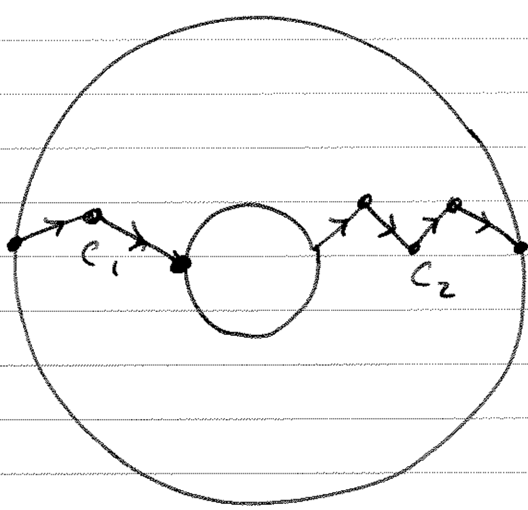
$$C = \#ABC - \#ACB = 3 - 3 = 0$$

$$I_1 = \#AB - \#BA \text{ outside } = 2 - 1 = 1$$

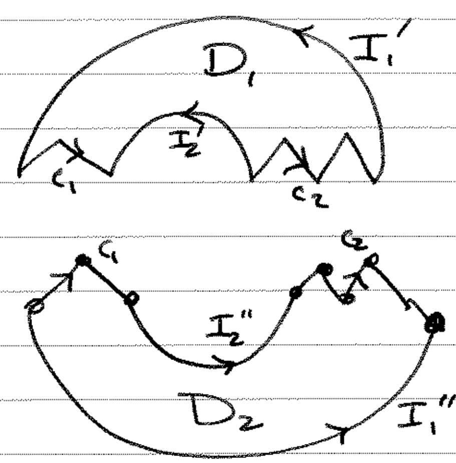
$$I_2 = \#AB - \#BA \text{ inside } = 1 - 0 = 1$$

So $C = I_1 - I_2$ for the annulus.

In general, let c_1 and c_2 be edges or sequences of edges which connect the inner and outer circles, as below:



These split the annulus into two cells



$I_1 = I_1' + I_1''$

$I_2 = I_2' + I_2''$

Then the Index Lemma applies to each cell, so we get

$$\left.\begin{aligned} \text{Content }(D_1) &= I_1' + I(c_1) - I_2' + I(c_2) \\ \text{Content }(D_2) &= I_1'' - I(c_2) - I_2'' - I(c_1) \end{aligned}\right\} \text{ where } I(c_1), I(c_2) \text{ is the index along } c_1 \text{ and } c_2.$$

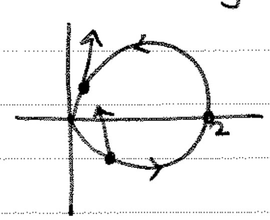
So

$$\left.\begin{aligned} \text{Content (Annulus)} &= \text{Content }(D_1) + \text{Content }(D_2) \\ &= I_1' - I_2' + I_1'' - I_2'' \\ &= I_1 - I_2 \end{aligned}\right\} \text{ since } I(c_1) \text{ and } I(c_2) \text{ cancel.}$$

§8

① $V\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}y\\1-x^2\end{pmatrix}$ points due north when $y=0$ and $1-x^2>0$.

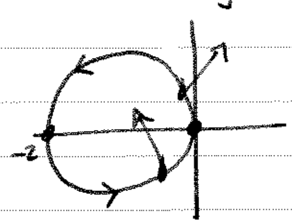
(a) On $x^2+y^2=2x$ this requires $x=0$ or 2 and $1-x^2>0$, i.e. $x=y=0$.



Before $\begin{pmatrix}0\\0\end{pmatrix}$, $y>0$, so V points in region A, and after $\begin{pmatrix}0\\0\end{pmatrix}$, $y<0$, so V points into region B.

Hence $W=+1$.

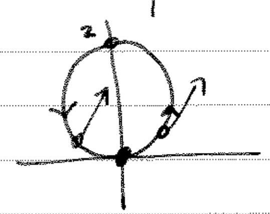
(b) On $x^2+y^2=-2x$, again $\begin{pmatrix}0\\0\end{pmatrix}$ is the only place V points north.



Before $\begin{pmatrix}0\\0\end{pmatrix}$, $y<0$, and after $\begin{pmatrix}0\\0\end{pmatrix}$, $y>0$, so we have a BA transition.

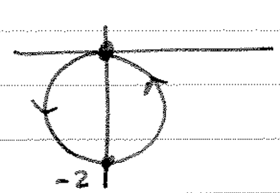
$W=-1$

(c) On $x^2+y^2=2y$, $y=0$ gives $x=0$ so $\begin{pmatrix}0\\0\end{pmatrix}$ is again the only north pointing point. Before $\begin{pmatrix}0\\0\end{pmatrix}$, and after $\begin{pmatrix}0\\0\end{pmatrix}$, $y>0$, so we have an AA non-transition.



$W=0$

(d) On $x^2+y^2=-2y$, similarly, $\begin{pmatrix}0\\0\end{pmatrix}$ is the only point where V points north, and $y<0$ both before and after this, so

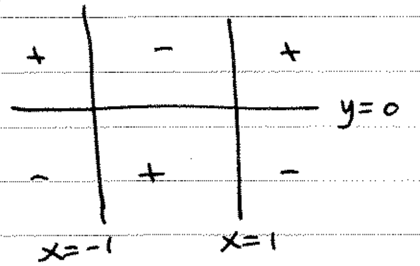


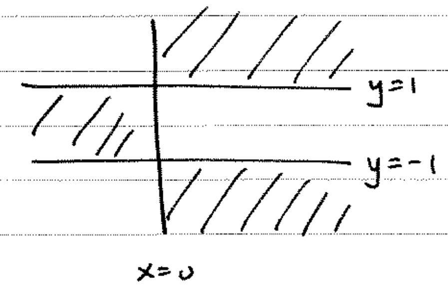
$W=0$.

② $v\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}y(x^2-1)\\x(y^2-1)\end{pmatrix}$ points due north when

$\left(y=0 \text{ or } x=\pm 1\right)$ and $x(y^2-1)>0$ (shaded below)

$\left\{\begin{array}{l}x\text{-coord}\\ \text{of } v \text{ is}\end{array}\right\}$ $\left\{\begin{array}{l}0 \text{ along}\\ \text{these lines,}\\ \text{sign as}\\ \text{indicated}\end{array}\right\}$

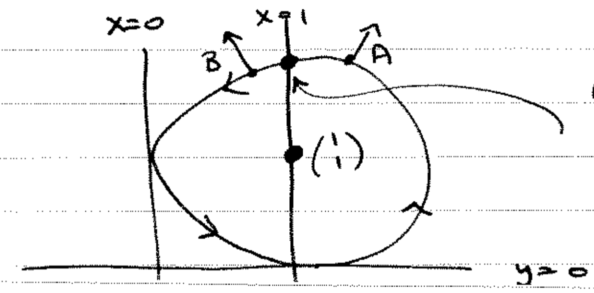




$\left\{\begin{array}{l}y\text{-coord}\\ \text{of } v\end{array}\right\}$ (pos where shaded)

(a) $x^2+y^2-2x-2y+1=0$

$(x-1)^2+(y-1)^2=1$

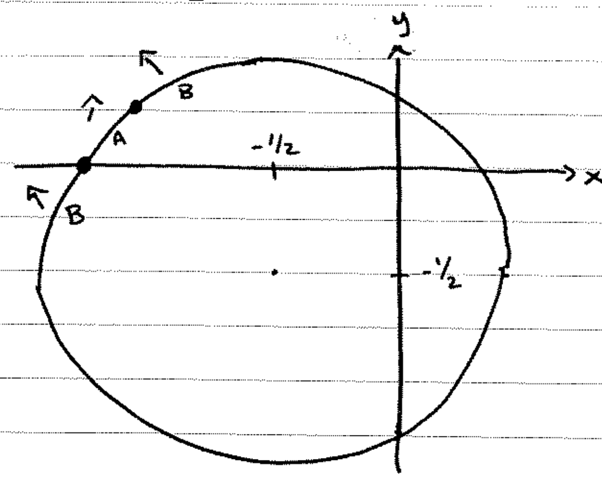


One due north point $\binom{1}{2}$. AB so

$W=1$

(b) $x^2+y^2+x+y=\frac{1}{2}$

$(x+\frac{1}{2})^2+(y+\frac{1}{2})^2=1$



North pointing points:

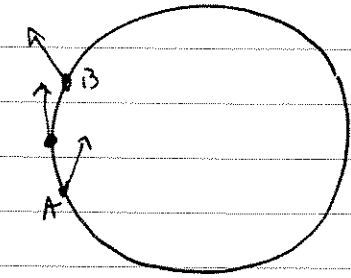
$\begin{pmatrix}-1\\(\sqrt{3}-1)/2\end{pmatrix}$ B to A

$\begin{pmatrix}(-\sqrt{3}-1)/2\\0\end{pmatrix}$ A to B

$W=0$

(c) $\binom{-1}{0}$ only

$W=-1$



(d) $\binom{+1}{\pm\sqrt{3}}$ and $\binom{-2}{0}$

$W=3$