Homework #4 §4 #2, 3, 6, 9 ($\mathcal{D}$, $\mathcal{I}$, $\mathcal{F}$) W'10

② a) If $\exists P$ such that $P \leftarrow A$ then $A \neq \emptyset$.

Proof: let N be a nhood of P. Then $N \cap A \neq \emptyset$ so $A \neq \emptyset$. //

b) $A \subseteq B$ and $P \leftarrow A \Rightarrow P \leftarrow B$

Proof: Let N be a nhood of P. Then $N \cap A \neq \emptyset$ and $N \cap B \supseteq N \cap A$ so $N \cap B \neq \emptyset$. Hence $P \leftarrow B$. //

c) If $P \leftarrow A \cup B$ then $P \leftarrow A$ or $P \leftarrow B$.

Proof: We'll prove the equivalent statement: if $P \leftarrow A \cup B$ and $P \not\leftarrow A$ then $P \leftarrow B$. So, suppose $P \leftarrow A \cup B$ and $P \not\leftarrow A$. Then some nhood N_1 of P satisfies $N_1 \cap A = \emptyset$. For any nhood N of P, there is a nhood $N_2 \subseteq N \cap N_1$ with $P \in N_2$. Since $P \leftarrow A \cup B$, $N_2 \cap (A \cup B) \neq \emptyset$. Now $N_2 \cap (A \cup B) = (N_2 \cap A) \cup (N_2 \cap B)$ and $N_2 \cap A \subseteq N_1 \cap A = \emptyset$, so $N_2 \cap A = \emptyset$ and $N_2 \cap B \neq \emptyset$. Then $N \cap B \supseteq N_2 \cap B$, so every nhood of P contains a point of B, hence $P \leftarrow B$. //

③ In $\mathcal{D}$, $x \leftarrow A \iff x \in A$.

Proof: Certainly $x \in A \Rightarrow x \leftarrow A$. Now if $x \leftarrow A$, the nhood $\{x\}$ of x must intersect A, so $x \in A$. //

In $\mathcal{I}$, $x \leftarrow A \iff A \neq \emptyset$.

Proof: If $A = \emptyset$ then $N \cap A = \emptyset$ so $x \leftarrow A$ is never true. If $x \leftarrow A$ then the nhood X of x satisfies $X \cap A \neq \emptyset$, i.e. $A \neq \emptyset$. //

⑥ (a) A set A is open iff each $P \in A$ has a nhood $N \subseteq A$.

Proof: If A is open then $P \in A \Rightarrow p \not\leftarrow X \setminus A$. Hence $\exists$ a nhood N of p such that $N \cap (X \setminus A) = \emptyset$. Therefore $N \subseteq A$.

Conversely, if p has a nhood $N \subseteq A$ then $N \cap (X - A) = \emptyset$, so $p \not\leftarrow \sim A$. //

(b) Any union of open sets is open.

Proof: Let U_α be open for each α. If $p \in \bigcup_\alpha U_\alpha$ then $p \in U_\alpha$ for some α. Then p has a nhood $N \subseteq U_\alpha$, so $N \subseteq \bigcup_\alpha U_\alpha$.

(c) A finite intersection of open sets is open.

Proof: Use induction on the number of open sets. For one set the result is trivially true. Suppose true for $n-1$ open sets. If $U_1, \ldots, U_n$ are open, $U_1 \cap \cdots \cap U_{n-1}$ is open by inductive hypothesis, so it suffices to show that the intersection of two open sets is open. If $p \in U_1 \cap U_2$ then $p \in U_1$ and $p \in U_2$, so there are nhoods N_1 and N_2 satisfying $N_i \subseteq U_i$ and $p \in N_1 \cap N_2$. Hence there is a nhood N of p with $N \subseteq N_1 \cap N_2 \subseteq U_1 \cap U_2$. Thus $U_1 \cap U_2$ is open. //

(d) Infinite intersections of open sets may not be open:

$$\bigcap_{n=1}^{\infty} \left(-\tfrac{1}{n}, 1+\tfrac{1}{n}\right) = [0,1]$$

But they may be: $$\bigcap_{n=1}^{\infty} (-n, n) = (-1,1).$$

(9) For $\mathcal{D}$, $\mathcal{I}$ and $\mathcal{F}$.

(a) $\{x\}$ closed $\forall x \in X$ (b) S^1 closed (c) Compact $\Rightarrow$ closed
(d) Compact $\Rightarrow$ bounded

$\mathcal{D}$: $x \leftarrow A \Leftrightarrow x \in A$ by (3)
All sets are closed, since $\mathcal{D} = \mathcal{P}(X)$.
Compact $\Leftrightarrow$ finite: Proof: Clearly finite $\Rightarrow$ compact. Conversely, if $c_1, c_2, \ldots$ are distinct elements of C then no point is near the sequence $(c_1, c_2, \ldots)$.

So

(a) T (b) T (c) T (d) T (All true.)

$\mathcal{I}$: $x \leftarrow A \Leftrightarrow A \neq \phi$ by (3)
A closed $\Leftrightarrow$ $A = \phi$ or $A = X$, since $\mathcal{I} = \{X\}$.
Every set is compact. Proof: if $(c_1, c_2, \cdots)$ is a sequence in C and $c \in C$ then $c \leftarrow \{c_k, c_{k+1}, \cdots\}$ for each k. //

So

(a) - (d) are all False

(9) $\mathcal{F}$ = finite topology

Nhoods $N = X \setminus \{x_1, \dots, x_n\}$ for some finite subset $\{x_1, \dots, x_n\} \subseteq X$.
Hence

$$(*) \begin{cases} x \leftarrow A & \iff \forall \text{ nhoods } N \text{ of } x,\ N \cap A \neq \emptyset \\ & \iff \text{if } x \in X \setminus \{x_1, \dots, x_n\} \text{ then } A \cap X \setminus \{x_1, \dots x_n\} \neq \emptyset \\ & \iff \text{if } x \notin \{x_1, \dots, x_n\} \text{ then } A \nsubseteq \{x_1, \dots, x_n\} \end{cases}$$

and I claim that this is equivalent to

$$x \in A \quad \text{or} \quad A \text{ is infinite.}$$

<u>Proof</u> That $x \leftarrow A \iff (x \in A$ or A is infinite$)$:

($\Leftarrow$) Certainly $x \in A \Rightarrow x \leftarrow A$. Now suppose A is infinite. Then $A \nsubseteq \{x_1, \dots x_n\}$ for any finite set $\{x_1, \dots, x_n\}$, so $x \leftarrow A$ by $(*)$.

($\Rightarrow$) We will show $x \leftarrow A$ and A finite $\Rightarrow x \in A$, which is equivalent. Write $A = \{x_1, \dots, x_n\}$. If $x \in A$ we're done. Otherwise $X \setminus A$ is a nhood of x which does not meet A, which contradicts $x \leftarrow A$. //

<u>Summary</u>: For A finite, $x \leftarrow A \iff x \in A$.
For A infinite, $x \leftarrow A$ always. (i.e. for all $x \in X$)

Now U open $\iff X \setminus U$ closed so closed $\iff$ finite.

<u>All</u> subsets are compact: If C is finite then any sequence $(c_1, c_2, \dots)$ in C must repeat some element infinitely often, and that element is near the sequence. If C is infinite then any $c \in C$ is near any sequence $(c_1, c_2, \dots)$ with an infinite number of distinct terms.

<u>So</u> (a) True (b), (c), (d) False.