Homework 3 Math 5520 W'10

§3 #3, 4, 7, 8, 10

(3)

Set	Bounded	Closed	Cpt
(a) $\{x \in \mathbb{R}^2 \mid \|x\| \le 1\}$	Yes	Yes	Yes
(b) $\{x \in \mathbb{R}^2 \mid \|x\| = 1\}$	Yes	Yes	Yes
(c) $\{x \in \mathbb{R}^2 \mid \|x\| < 1\}$	Yes	No	No
(d) $\mathbb{R} \times 0$	No	Yes	No
(e) $\mathbb{R}^2 - \mathbb{R} \times 0$	No	No	No
(f) $\mathbb{Z}^2$	No	Yes	No
(g) $\{(\frac{1}{n}, 0) \mid n = 1, 2, 3, \ldots\}$	Yes	No	No
(h) $\mathbb{R}^2$	No	Yes	No
(i) ϕ	Yes	Yes	Yes

(4) Neither closed nor bounded is preserved by homeomorphism

In Homework (1) we showed $(0,1) \cong \mathbb{R}$. $(0,1)$ is not closed in $\mathbb{R}$ but $\mathbb{R}$ is. This example also works for boundedness.

(7) If X and Y are connected and $X \cap Y \neq \phi$ then $X \cup Y$ is connected.

Proof: If $X \cup Y = A \cup B$ is a disconnection, then $X = (X \cap A) \cup (X \cap B)$ would be a disconnection of X unless one of $X \cap A$ and $X \cap B$ is empty. Similarly for $Y \cap A$ and $Y \cap B$. Since both A and B are non-empty, $X \subseteq A$ and $Y \subseteq B$ or vice versa. But then any $x \in X \cap Y$ is in both A and B, so $A \cup B$ is not a disconnection after all. //

(8) If $C \subseteq \mathbb{R}$ is connected, then C is an interval.

Proof: If $a, b \in C$ and $a < c < b$ then $C = \big(C \cap (-\infty, c)\big) \cup \big(C \cap (c, \infty)\big)$ would disconnect C if $c \notin C$. To see that this implies C is an interval, we must consider cases as on the next page.

Comments on intervals: Let C be a nonempty set of reals, and

Suppose $C \subseteq \mathbb{R}$ satisfies

$$a, b \in C \text{ and } a < c < b \Rightarrow c \in C.$$

Let

$$m = \text{glb } C \quad \text{and} \quad M = \text{lub } C.$$

Then one of the following holds:

(1) If both m and M are infinite ($m = -\infty$ and $M = \infty$) then $C = \mathbb{R}$.
(2) m infinite and M finite implies $C = (-\infty, M)$ or $(-\infty, M]$.
(3) m finite and M infinite implies $C = (m, \infty)$ or $[m, \infty)$.
(4) Both m and M finite implies $C = (m, M)$, $(m, M]$, $[m, M)$ or $[m, M]$.

Proof: C nonempty $\Rightarrow$ (1)-(4) exhaust the possibilities for m and M.
Case (1): given $x \in \mathbb{R}$ $\exists$ $a, b \in C$ with $a < x$ and $x < b$ so $x \in C$.
Case (2): given $x \in \mathbb{R}$ with $x < M$, $\exists$ $a, b \in C$ with $a < x < b$. Thus $x \in C$. Hence $(-\infty, M) \subseteq C$. $M = \text{glb } C \Rightarrow (M, \infty) \subseteq \mathbb{R} \setminus C$ so $C = (-\infty, M)$ or $(-\infty, M]$.
Case (3): Like (2).
Case (4): given $x \in (m, M)$, $x > m = \text{glb } C \Rightarrow \exists\, a \in C$, $a < x$. Similarly, $x < M = \text{lub } C \Rightarrow \exists\, b \in C$, $b > x$. Then $a < x < b$ so $x \in C$. Hence

$$(m, M) \subseteq C \subseteq [m, M]$$ as claimed. //

⑩ Suppose that for each pair of points $x, y \in S$, there is a connected subset $C \subseteq S$ containing both x and y. Then S is connected.

Proof: Suppose S is disconnected and that $S = A \cup B$ is a disconnection. Thus, no point of A is near B and vice versa. Let $a \in A$, $b \in B$ and suppose $a, b \in C \subseteq S$. Then

$$C = (C \cap A) \cup (C \cap B)$$

with $a \in C \cap A \neq \phi$ and $b \in C \cap B \neq \phi$. Further, no element of $C \cap A$ is near $C \cap B$ since it would then be near B, and vice versa. Hence C is disconnected.

It follows that any subset containing both a and b must be disconnected. //