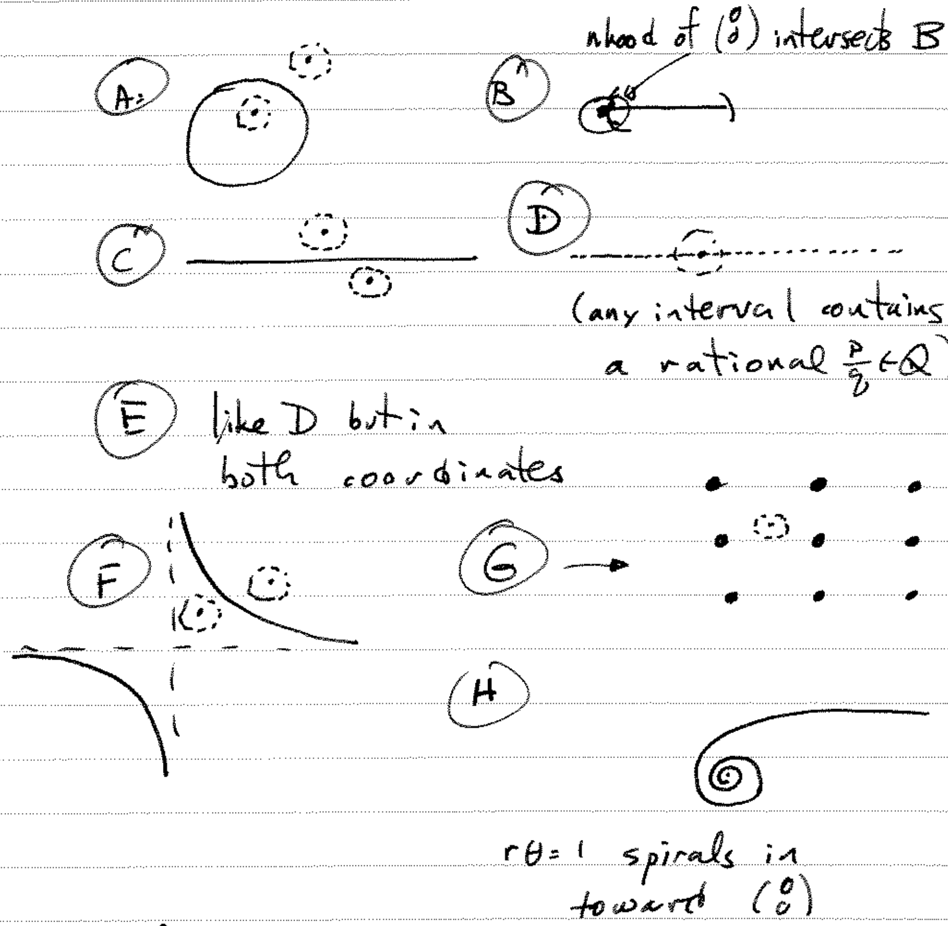
Math 5520 W'10 HW 2 Solutions

§2 #1, 3, 5, 8, 9

①

| Set | Near points |
| --- | --- |
| $A = S^1$ | $A$ |
| $B = (0,1) \times \{0\}$ | $[0,1] \times \{0\}$ |
| $C = \mathbb{R} \times \{0\}$ | $C$ |
| $D = \mathbb{Q} \times \{0\}$ | $C$ |
| $E = \mathbb{Q}^2$ | $\mathbb{R}^2$ |
| $F = \{\binom{x}{y} \mid xy = 1\}$ | $F$ |
| $G = \mathbb{Z}^2$ | $G$ |
| $H = \{\binom{r\cos\theta}{r\sin\theta} \mid r\theta = 1\}$ | $H \cup \{\binom{0}{0}\}$ |
| $I = \emptyset$ | $I$ |




I: Here is my picture of the empty set:

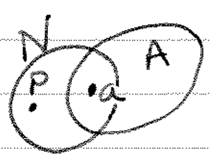
③ If $A$ has a point near it but not in it, then $A$ is infinite.

Proof 1: This is equivalent to: If $A$ is finite then $P \leftarrow A \Rightarrow P \in A$. So suppose $A = \{a_1, \dots, a_n\}$ and $P \notin A$. Let $d = \frac{1}{2} \min \|P - a_i\|$. Then $D_d(P) \cap A = \emptyset$ since $\|P - a_i\| > d$ for each $a_i$. Thus $P \not\leftarrow A$. //

Proof 2: Let $P \leftarrow A$, $P \notin A$. Let $a_1 \in A \cap D_1(P)$. Then let $a_2$ be in $A$ and less than half as far from $P$ as $a_1$, let $a_3$ be in $A$ and less than half as far from $P$ as $a_2$, etc. Then all the $a_i$'s are different so $A$ is infinite. (Note $a_i \neq P$ since $P \notin A$ so $\frac{1}{2} d(P, a_i)$ is still positive, so picking $a_{i+1}$ is possible.) //

(5) If every point of A is near B then $P \leftarrow A \Rightarrow P \leftarrow B$.

Proof: Suppose every point of A is near B and $P \leftarrow A$. Let N be a nhood of P. Then $\exists\, a \in N \cap A$ since $P \leftarrow A$. Now N is also a nhood of a, which is in A, hence near B. So $N \cap B$ is nonempty. So, every nhood of P contains a point of B and thus $P \leftarrow B$. //




(8) If f is "near-continous" then f is "usual-continuous". Equivalenty, if f is not "usual-continuous" then f is not "near-continuous". Suppose $f: A \longrightarrow B$.

Proof: Usual continuous $\Leftrightarrow \forall P \in A\ \forall \varepsilon > 0\ \exists \delta > 0$ s.t. $f(D_\delta(P)) \subseteq D_\varepsilon(f(P))$

So Not usual continuous $\Leftrightarrow \exists P \in A\ \exists \varepsilon > 0\ \forall \delta > 0\ \ f(D_\delta(P)) \not\subseteq D_\varepsilon(f(P))$.

Thus, for each $\delta > 0$ we can find an $a_\delta \in D_\delta(P)$ but $f(a_\delta) \notin D_\varepsilon(f(P))$

[Note: We are using the P & ε which are asserted to exist by the assumption that f is not "usual-continuous".]

Then $P \leftarrow \{a_\delta \mid \delta > 0\}$ since any nhood N of P contains some $D_\delta(P)$.

But $f(P) \not\leftarrow f\{a_\delta \mid \delta > 0\}$ since each $f(a_\delta)$ is outside the nhood $D_\varepsilon(f(P))$. //

(9) If f is "usual-continuous" then f is "near-continuous".

Proof: Suppose $P \leftarrow A$ and f is "usual-continuous". That is, if N is a nhood of $f(P)$ then there exists a nhood $U$ of P with $f(U) \subseteq N$. Now $U$ is a nhood of P, so $\exists\, a \in U \cap A$. Hence $f(a) \in f(U) \cap f(A) \subseteq N \cap f(A)$. Thus, every nhood of $f(P)$ contains a point of $f(A)$ and so $f(P) \leftarrow f(A)$. //