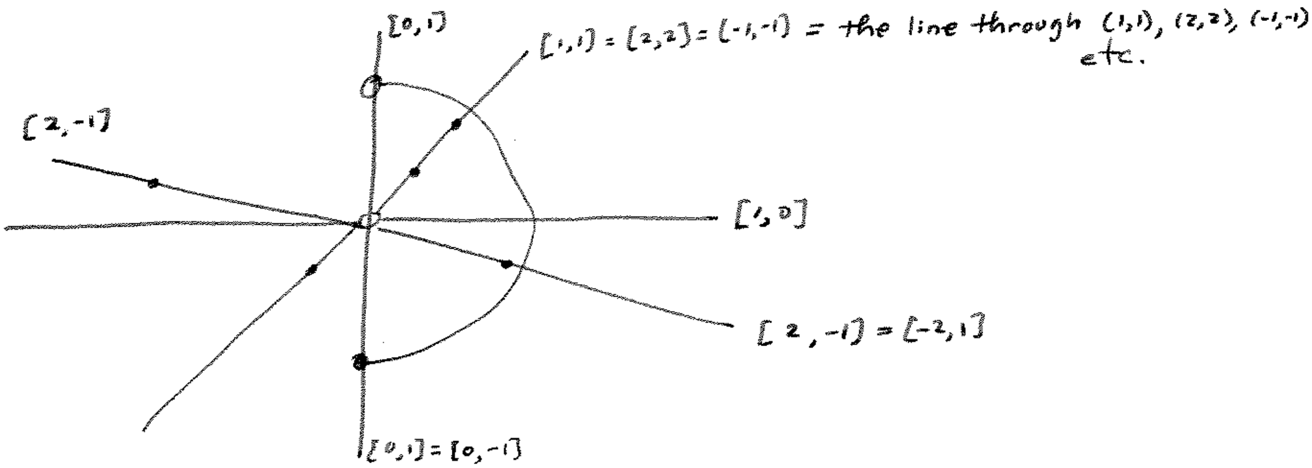
Projective Spaces

The <u>projective line P^1</u> is the set of possible slopes in the plane, which we can write

$$P^1 = \mathbb{R}P^1 = \mathbb{R}^2 - (0,0) \,/\!\sim \qquad \text{where } (x,y) \sim (rx,ry) \text{ if } r \neq 0$$

since (x,y) and (rx,ry) determine the same line.



Write $[x,y]$ for the equivalence class of (x,y), that is

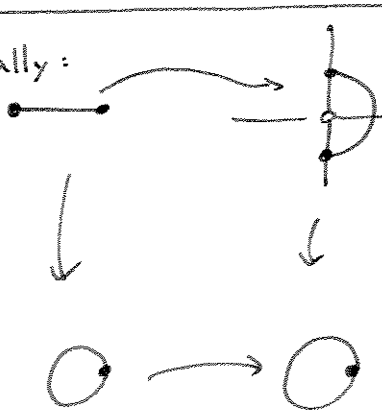
$$[x,y] = \text{the line through } (0,0) \text{ and } (x,y)$$

The right half of the unit circle passes through every line, and is one-to-one except at the two ends, which are joined, hence

$$\mathbb{R}P^1 \cong S^1$$

$$\begin{tikzcd} I \arrow[r] \arrow[d] & \mathbb{R}^2 - 0 \arrow[d] \\ S^1 \arrow[r, "\cong"] & \mathbb{R}P^1 \end{tikzcd}$$

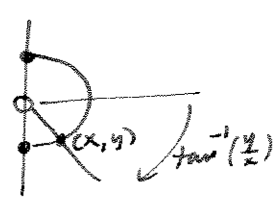
pictorially:

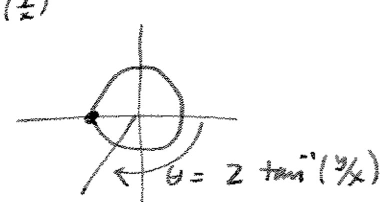


A specific formula:

$$\mathbb{R}P^1 \longrightarrow S^1$$

$$[x,y] \longmapsto e^{i\theta}, \quad \theta = 2\tan^{-1}\left(\tfrac{y}{x}\right) \text{ if } x \neq 0, \quad \theta = \pi \text{ if } x = 0.$$





IE $[x,y] \mapsto (x+iy)^2 / \|(x+iy)^2\|$

The projective line can be described as $\mathbb{R} \cup \{\infty\}$ as follows. We can embed $\mathbb{R}$ into $\mathbb{RP}^1$, missing only one point by homeomorphisms

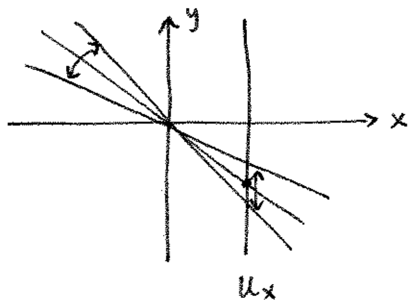
$$\begin{array}{rcll} \mathbb{R} & \longrightarrow & \mathcal{U}_x = \{[x,y] \mid x \neq 0\} & \\ y & \longmapsto & [1,y] & \\ y/x & \longleftarrow & [x,y] & (x \neq 0) \end{array} \qquad \text{and} \qquad \begin{array}{rcll} \mathbb{R} & \longrightarrow & \mathcal{U}_y = \{[x,y] \mid y \neq 0\} & \\ x & \longmapsto & [x,1] & \\ x/y & \longleftarrow & [x,y] & (y \neq 0) \end{array}$$

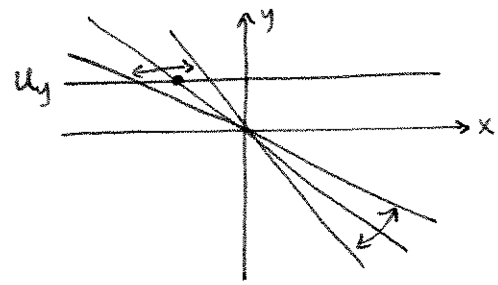
with inverses

Then

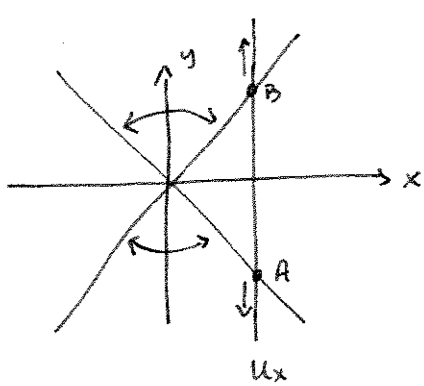
$$\begin{aligned} \mathbb{RP}^1 = \mathcal{U}_x \cup \{[0,1]\} &= \mathcal{U}_y \cup \{[1,0]\} \\ \cong \mathbb{R} \cup \{\infty\} \quad &\cong \mathbb{R} \cup \{\infty\} \end{aligned}$$

Here, a nhood of lines, thought of as points of $\mathbb{RP}^1$, translates into an ordinary nhood, i.e. interval, in $\mathbb{R}$ for points of $\mathcal{U}_x$ or $\mathcal{U}_y$,

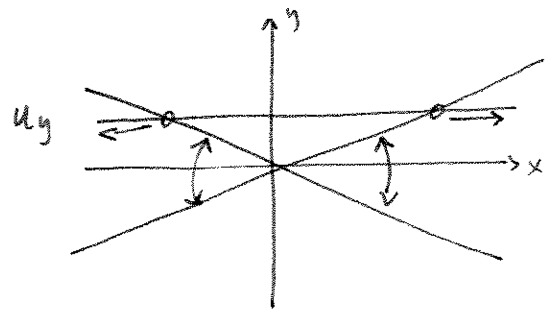




while nhoods of the exceptional points translate into nhoods of ∞ of the form $(-\infty, A) \cup (B, \infty) \cup \{\infty\}$



a nhood of the y-axis,
i.e. of $[0,1]$ in $\mathbb{RP}^1$



a nhood of the x-axis,
i.e. of $[1,0]$ in $\mathbb{RP}^1$

The projective plane $\mathbb{RP}^2$ is the space of lines through the origin in $\mathbb{R}^3$,

$$\mathbb{RP}^2 = \mathbb{R}^3 - (0,0,0)/\sim \qquad \text{where} \quad (x,y,z) \sim (rx,ry,rz) \quad \text{if } r \neq 0,$$

since (x,y,z) and (rx,ry,rz) determine the same line through the origin. Write

$$[x,y,z] = \text{the line through } (0,0,0) \text{ and } (x,y,z)$$

for the equivalence class of (x,y,z). That is

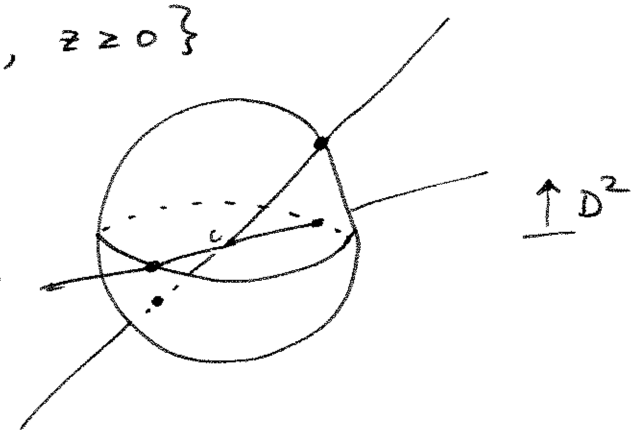
$$[x,y,z] = \{(rx,ry,rz) \mid r \neq 0\}.$$

Every such line intersects the sphere $S^2 = \{(x,y,z) \mid x^2+y^2+z^2=1\}$ in exactly two points, so

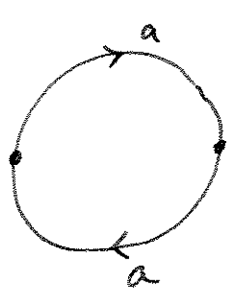
$$\mathbb{RP}^2 = S^2/\sim \qquad \text{where} \quad p \sim \pm p, \text{ for } p \in S^2,$$

and, in fact, each such line hits S^2 in a point at which $z \geq 0$. That is, each such line hits the disk which is the upper hemisphere of S^2:

$$D^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1, \quad z \geq 0\}$$



If the line is not horizontal, it hits this D^2 in one interior point (where $z > 0$ and $x^2+y^2 < 1$) while if the line is horizontal, it hits D^2 in two boundary points opposite one another. Thus, $\mathbb{RP}^2$ is the quotient of D^2 with plane model



If we let $\quad \mathcal{U}_x = \{[x,y,z] \mid x \neq 0\}$

$\mathcal{U}_y = \{[x,y,z] \mid y \neq 0\}$

and $\quad \mathcal{U}_z = \{[x,y,z] \mid z \neq 0\}$

then we have homeomorphisms

$$\mathbb{R}^2 \longrightarrow \mathcal{U}_x \qquad \mathbb{R}^2 \longrightarrow \mathcal{U}_y \qquad \mathbb{R}^2 \longrightarrow \mathcal{U}_z$$

$$(y,z) \longmapsto [1,y,z] \qquad (x,z) \longmapsto [x,1,z] \qquad (x,y) \longmapsto [x,y,1]$$

with inverses

$$(y/x, z/x) \longleftarrow [x,y,z] \qquad (x/y, z/y) \longleftarrow [x,y,z] \qquad (x/z, y/z) \longleftarrow [x,y,z].$$

and the complements of $\mathcal{U}_x, \mathcal{U}_y, \mathcal{U}_z$ are copies of $\mathbb{R}P^1$,

$$\mathbb{R}P^2 - \mathcal{U}_x = \{[0,y,z]\} \cong \mathbb{R}P^1 \quad , \quad \mathbb{R}P^2 - \mathcal{U}_y = \{[x,0,z]\} \cong \mathbb{R}P^1$$

$$\text{and} \qquad \mathbb{R}P^2 - \mathcal{U}_z = \{[x,y,0]\} \cong \mathbb{R}P^1$$

commonly called "the circle at infinity" for the corresponding $\mathbb{R}^2$, since, for example, in $\mathcal{U}_z$ as $z \to 0$ with x and y fixed,

$$[x,y,z] \mapsto (x/z, y/z) \text{ "approaches infinity".}$$

Using these homeomorphisms, we can embed geometry in $\mathbb{R}^2$ into $\mathbb{R}P^2$ and study how it interacts with the corresponding circle at infinity.

For example, the line $y = mx + b$ under the homeomorphism $\mathcal{U}_z \longrightarrow \mathbb{R}^2$ turns into

$$\left(\frac{y}{z}\right) = m\left(\frac{x}{z}\right) + b$$

$$\text{OR} \qquad y = mx + bz.$$

The solutions to this equation in $\mathbb{R}P^2$ are the points $[x, mx+bz, z]$, In U_z, $z \neq 0$, so these points are $[x, mx+bz, z] = [x/z, m(x/z)+b, 1]$ corresponding to $y = mx+b$ in $\mathbb{R}^2$. The corresponding circle at ∞ has $z=0$, so we get solutions

$$[x, mx, 0] = [1, m, 0],$$

that is, the single point "at infinity" corresponding to slope $y/x = m/1 = m$. Note that the intersection of $y = mx + bz$ with the circle at infinity depends only on m, not on b, hence

> In $\mathbb{RP}^2$, all lines of slope m in U_z intersect
>
> at $[1, m, 0]$ in the circle at ∞.

This is paraphrased by saying "parallel lines intersect at infinity", a rather mysterious way of describing the calculation we have just done.

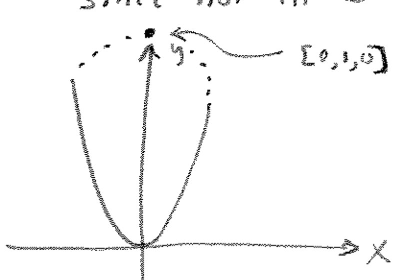
Note also that the lines of infinite slope in $\mathbb{R}^2$, $x = a$, correspond to $x/z = a$, $x = az$, in U_z, and that this line intersects the circle at infinity $z=0$ in the point $[0,1,0]$ corresponding to "infinite slope" $1/0$. Alternatively,

$$\lim_{m \to \pm\infty} [1, m, 0] = \lim_{m \to \pm\infty} [1/m, 1, 0] = [0,1,0].$$

<u>A quadratic example</u>: consider $y = x^2$ in $\mathbb{R}^2 \cong U_z$. This becomes

$$\frac{y}{z} = \left(\frac{x}{z}\right)^2 \qquad \text{OR} \qquad \frac{y}{z} = \frac{x^2}{z^2} \qquad \text{or} \qquad yz = x^2 .$$

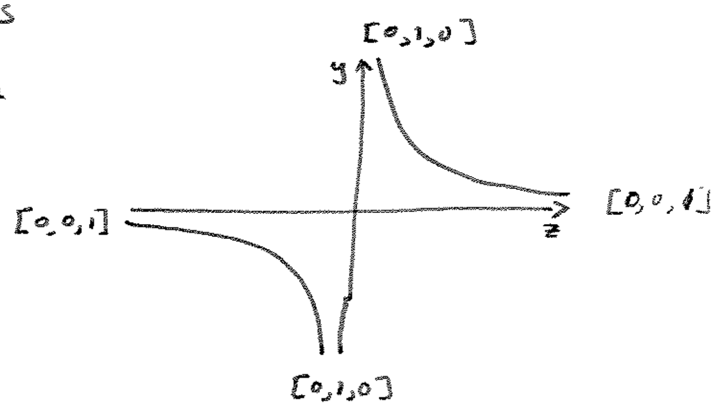
Intersecting with the circle at infinity $z=0$, we get $x^2 = 0$, so $x = 0$. Since not all 3 coordinates can be 0, $y \neq 0$, and $[0, y, 0] = [0,1,0]$,



i.e. the parabola $y = x^2$ intersects the circle at infinity in the point corresponding to infinite slope.

Another quadratic example: consider the hyperbola $y = 1/z$ in $\mathbb{R}^2 \cong U_x$. In U_x this becomes

$$\frac{y}{x} = \frac{1}{z/x} = \frac{x}{z} \qquad \text{or } yz = x^2$$



If we intersect this with the circle at ∞ $x = 0$ we get $yz = 0$, so $y = 0$ or $z = 0$. If $y = 0$ we get $[0,0,1]$, and if $z = 0$ then we get $[0,1,0]$.

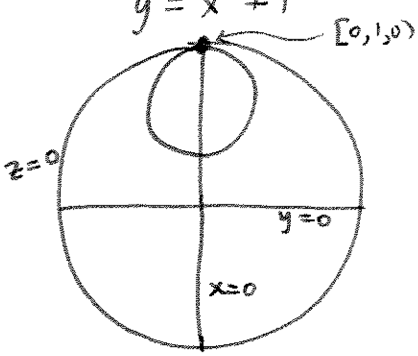
NOTE that in $\mathbb{R}P^2$, these two curves have become identical!

> In $\mathbb{R}P^2$ all quadratics become equivalent! The distinction between parabolas, hyperbolas and ellipses is simply a question of how they intersect the circles at infinity!

Final example: $\boxed{yz = x^2 + z^2}$

U_z $(z \neq 0)$

$y = x^2 + 1$



The parabola is tangent to the circle at infinity since $z = 0$ gives

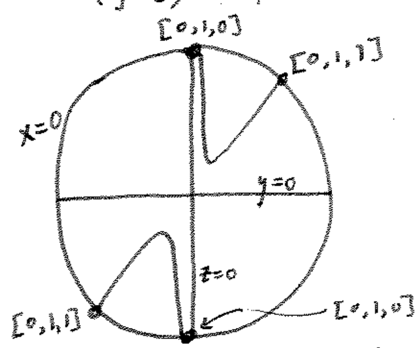
$0 = x^2$

which has $x = 0$ as a double root.

U_x $(x \neq 0)$

$yz = z^2 + 1$

$(y - z)z = 1$

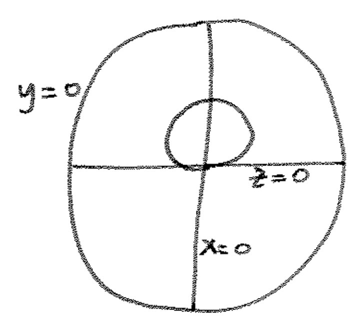


The hyperbola intersects the circle at infinity in two points: if $x = 0$ then $yz = z^2$ has solutions $z = 0$ $[0,1,0]$ and $y = z$ $[0,1,1]$.

U_y $(y \neq 0)$

$x^2 + z^2 = z$

$x^2 + (z - \frac{1}{2})^2 = \frac{1}{4}$



The circle (or ellipse) does not intersect the circle at infinity.