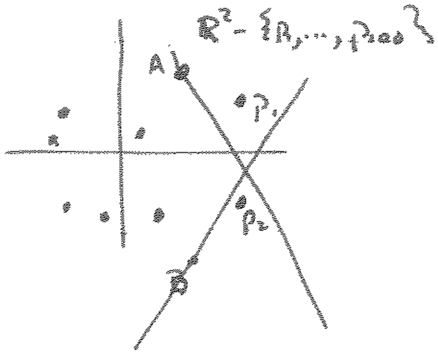
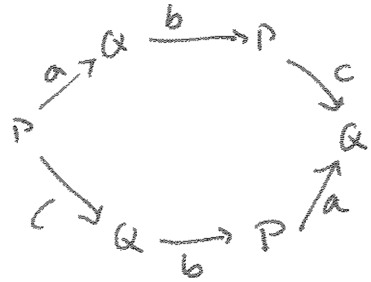
FINAL EXAM MS520 W09 Solutions



Let $A, B \in \mathbb{R}^2 - \{p_1, \dots, p_{100}\}$.
There are an infinite number of lines through A, ~~so~~ and at most 100 of them intersect $\{p_1, \dots, p_{100}\}$ so an infinite number of them lie in $\mathbb{R}^2 - \{p_1, \dots, p_{100}\}$. Similarly, there are an infinite number of lines through B which are in $\mathbb{R}^2 - \{p_1, \dots, p_{100}\}$. Hence one of these lines through A and one of these lines through B must intersect. This shows $\mathbb{R}^2 - \{p_1, \dots, p_{100}\}$ is path connected, and hence connected.

(I)



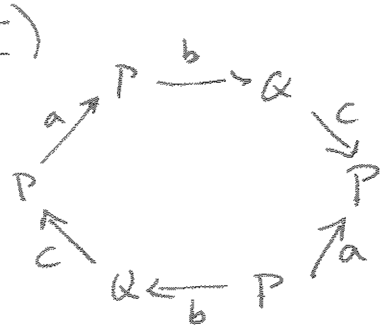
2 (a) $\chi = V - E + F = 2 - 3 + 1 = 0$

2 (b) Surface. (Each edge joins exactly one other. Vertices have cyclic orderings:

2 (c) $abca^{-1}b^{-1}c^{-1}$: no twisted pairs, so it is orientable

4 (d) Orientable and $\chi = 0 \Rightarrow$ Torus T

(II)



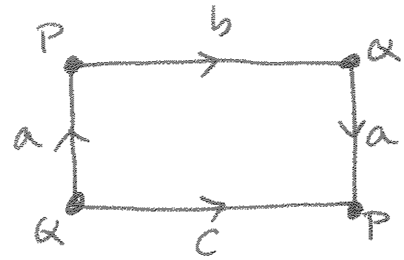
2 (a) $\chi = 2 - 3 + 1 = 0$

2 (b) Surface: each edge occurs twice & vertices are cyclic:

2 (c) $abca^{-1}bc$ has twisted pairs: Non-orientable.

4 (d) Klein bottle $K = P \# P$

$abac^{-1}$ $\qquad\qquad$ $dbdc^{-1}$



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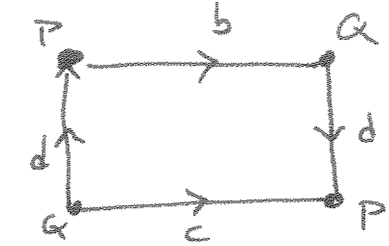

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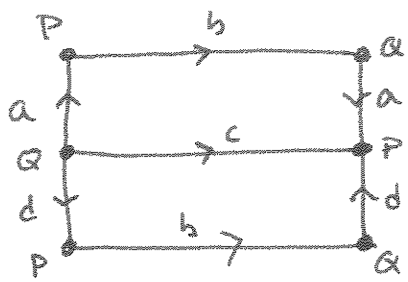


$= abad^{-1}b^{-1}d^{-1}$ (Erase c)

Twisted pair so non-orientable

$\chi = 2 - \cancel{4}\,3 + 1 = \cancel{3}\; 0$

Klein bottle $K = P^2 = P \# P$.

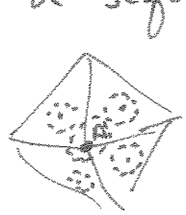
$(-1,1) \xrightarrow{f} (-1,0) \cup (0,1)$ cannot be continuous and onto ~~because~~ the continuous image of a connected set is connected.

5. $[-1,1] \xrightarrow{f} (-1,1)$ cannot be continuous and onto because the continuous image of a compact set is compact.

$(-4,1) \cup (1,4) \xrightarrow{f} (-1,1)$ by $f(x) = x/4$ is continuous and 1-1

7. $(-4,4) \to [-1,1]$ by $f(x) = x/4$ —— " —— .

8. Suppose there are an infinite number of triangles. Choose a point from the interior of each, ~~thi~~ and arrange them in a sequence. This sequence has no near point, because ~~the points have disjoint neigh~~ any other point A has a nhood in each triangle which does not intersect the point chosen in that triangle, and putting these together gives a nhood of A in the surface which does not contain any of these points. So the surface is not compact.

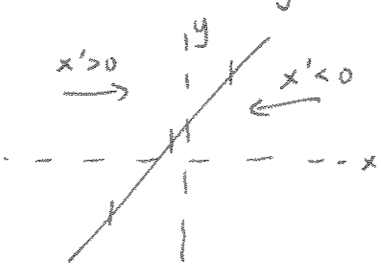


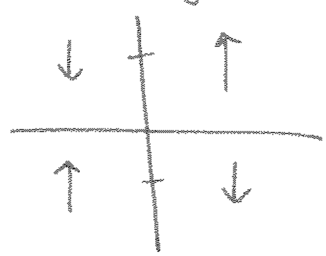
Hence compact $\Rightarrow$ finite number of triangles.

Conversely, if the number of triangles is compact then any sequence must have an infinite number of its terms in one of the triangles. Since triangles are compact, this subsequence has a point near it, and the near point is near the whole sequence as well. //

$$v\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y-x-1 \\ xy \end{pmatrix}$$

a) $x' = 0 \iff y = x+1$ $\qquad$ $y' = 0 \iff xy = 0$





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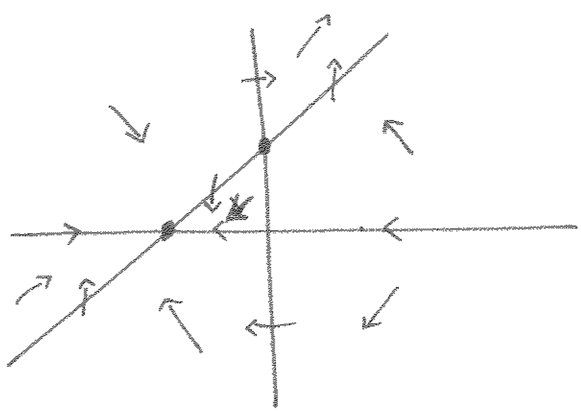
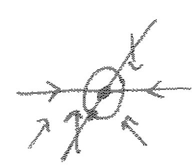
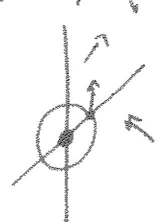


(b) Critical points are intersection of $x'=0$ with $y'=0$: $(-1,0)$ and $(0,1)$

(c) Poincaré method gives $\text{Index}(-1,0) = +1$



$\text{Index}(0,1) = -1$

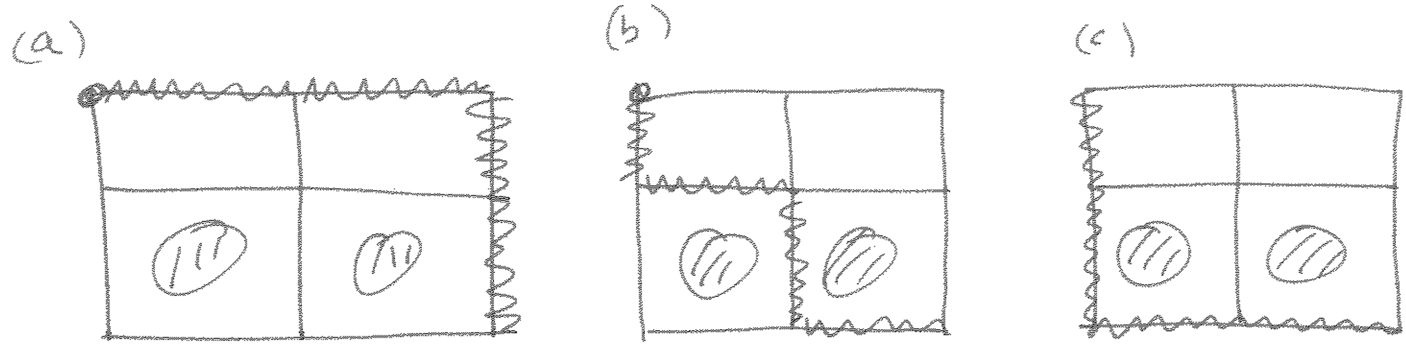


(d) $d\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \sqrt{2}$

$d\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 2$ $\quad$ so $\quad$ $W_r(\gamma) = \begin{cases} 0 & r < \sqrt{2} \\ -1 & \sqrt{2} < r < 2 \\ 0 & r > 2 \end{cases}$

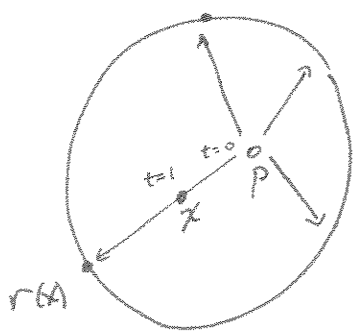
$-1+1$

0.



$\partial(\text{Outside face}) = (a) + (c)$ so $(a) \sim (c)$.

1. Suppose $f : D^2 \longrightarrow D^2$, $p \notin \operatorname{Im} f$ and $f(x) = x$ if $x \in \partial D^2$.



Let $r : D^2 - \{p\} \longrightarrow D^2$ send any x to the intersection of the ray

$$\{tx + (1-t)p \ : \ t > 0\}$$

with ∂D^2. Note $r(x) = x$ if $x \in \partial D^2$.

Then $rf : D^2 \to \partial D^2$ and $rf(x) = x$ if $x \in \partial D^2$.

Then let $R : \partial D^2 \to \partial D^2$ be the $90°$ rotation $R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$.

Then $Rrf : D^2 \to D^2$ is continuous but has no fixed point, which is impossible by the Brouwer fixed point theorem. It has no fixed points because interior points are sent to the boundary, hence not fixed, while boundary points are rotated $90°$, hence not fixed. //

2.
$$\begin{tikzcd} {[0,1]} \arrow[r, "\gamma"] \arrow[dr, "x\gamma"'] & \mathbb{R}^2 \arrow[d, "x"] \\ & \mathbb{R} \end{tikzcd}$$

$x\gamma$ is continuous so its image is a ~~compact~~ connected set in $\mathbb{R}$ which contains 1 but not 0. Hence it is a closed interval $[\varepsilon, M]$ with $0 < \varepsilon \le 1 \le M$.

Thus $\gamma(t)$ always has x coordinate $\geq \varepsilon$. //