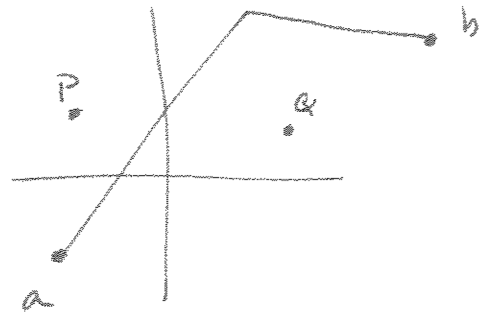
Math 5520 Midterm Solutions W'09

1)



Given any two points in $\mathbb{R}^2 - \{P,Q\}$, there is a path between them. (If the straight line path passes through P or Q, bend it slightly to avoid them.)

Hence every pair of points in $\mathbb{R}^2 - \{P,Q\}$ lies in a connected subset, and so $\mathbb{R}^2 - \{P,Q\}$ is connected. //

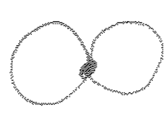
2) Suppose that $X \xrightarrow{f} Y$ is continuous and one-to-one, and that Y is discrete. Show that X is discrete.

Pf: Let $x \in X$. Then $\{f(x)\}$ is a nhood of $f(x)$ in Y. So $\{x\} = f^{-1}\{f(x)\}$ contains a nhood of x. Thus $\{x\}$ is a nhood in X. Hence X is discrete. //

3) Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous and all its values are irrational. Then f is constant.

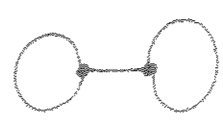
Pf: If $a, b \in \mathbb{R}$ with $f(a) \neq f(b)$, then every value between $f(a)$ and $f(b)$ must be in the image of $\mathbb{R}$, since $\mathbb{R}$ is connected. But there are rationals between any two reals, so this is impossible. Hence $f(a) = f(b)$ and so f is constant. //

4) a)



$\chi = V - E + F$
$= 1 - 2 + 0 = -1$

b)



$\chi = V - E + F$
$= 2 - 3 + 0 = -1$

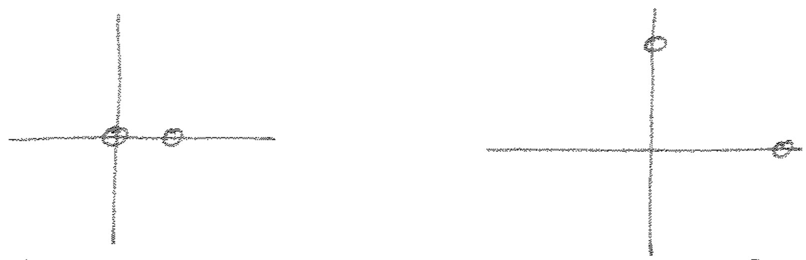
5) (a) $[0,\infty) \not\cong \mathbb{R}$ since $\mathbb{R}-\{p\}$ is disconnected for any $p\in\mathbb{R}$, while $[0,\infty)-\{0\}$ is connected.

(b) $S^1 \not\cong \mathbb{R}$ since S^1 is compact but $\mathbb{R}$ is not.

(c) $(0,1)\cup(1,2) \xrightarrow{f} (0,1)\cup(2,3)$ by $f(x)=\begin{cases} x & x<1 \\ x+1 & x>1\end{cases}$

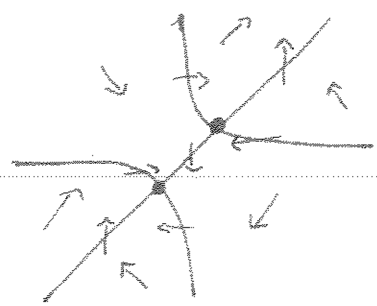
is a homeomorphism.

(d) $\mathbb{R}^2-\{(0,0),(1,0)\} \cong \mathbb{R}^2-\{(4,0),(0,4)\}$

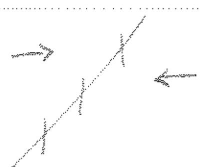


Rotate the 2^{nd}, holding $(0,4)$ fixed until $(4,0)$ is on the x-axis. Slide over until it coincides with $(0,0)$; shrink horizontally until the gap shrinks to 1.

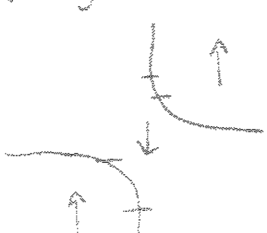
6) $V\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}y-x\\xy-1\end{pmatrix}$ (a)



Vertical on $y=x$

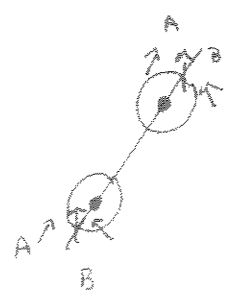


Horizontal on $xy=1$



(b) C.P. $(1,1)$, $(-1,-1)$

(c) $I=-1$ at $(1,1)$
$I=+1$ at $(-1,-1)$
by Poincaré method



(d) $W_\gamma=0$ until $(1,1)$ enters at $r=1$.
Then $W_\gamma=-1$ until $(-1,-1)$ also enters at $r=\sqrt5$.
Then $W_\gamma=0$ again for $r>\sqrt5$.

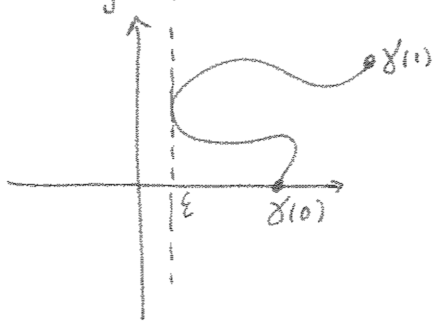
$$\begin{cases} 0 & 0<r<1 \\ -1 & 1<r<\sqrt5 \\ 0 & r>\sqrt5 \end{cases}$$

(7) Given $D^2 \xrightarrow{f} [-1,1]$ continuous. Let $g: D^2 \xrightarrow{f} [-1,1] \xrightarrow{i} D^2$ be if.

Then $\exists\,(x,y)$ s.t. $g(x,y) = (x,y)$ by Brouwer fixed point theorem. But $g(x,y) = if(x,y) = (f(x,y), 0)$ so $f(x,y) = x$. (and $y=0$, but that's another matter.)

Alternatively $[-1,1] \xrightarrow{i} D^2 \xrightarrow{f} [-1,1]$ must have a fixed point by the 1-dim Brouwer fixed point theorem. So $x = if(x) = f(x,0)$. //

(8) Let $\gamma : [0,1] \longrightarrow \mathbb{R}^2$ be a continuous curve which does not intersect the y-axis and which starts at $\gamma(0) = (1,0)$. Show that for some $\varepsilon > 0$, the image of γ lies in $\{(x,y) \mid x \geq \varepsilon\}$.



<u>Proof</u>: The composite $[0,1] \xrightarrow{\gamma} \mathbb{R}^2 \xrightarrow{\pi_1} \mathbb{R}$ must have a minimum value ε, since $[0,1]$ is compact. Since $\pi_1\gamma(0) = 1$, if $\varepsilon \leq 0$ then for some t in $[0,1]$, $\pi_1\gamma(t) = 0$, which is not possible by assumption, since $[0,1]$ is connected. Hence the minimum x-value is $\varepsilon > 0$. //