HW #5 §4 # 3, 7, 9*, 10, 11

* for the discrete, indiscrete, (co)finite & usual topologies

#3 In the discrete topology, subsets have no outside near points. Equivalently, $P \leftarrow A \Rightarrow P \in A$.

Proof: If $P \leftarrow A$, consider the nhood $\{P\}$. It must intersect A, and so $P \in A$. //

In the indiscrete topology, every point is near every nonempty set.
Proof: Suppose P is a point and A a nonempty subset. The only nhood of P is the whole space X, and $X \cap A = A \neq \phi$, so $P \leftarrow A$. //

#7 U is open $\Longleftrightarrow$ U^c is closed

Proof: U is open $\Longleftrightarrow$ $\forall P \in U$ $\exists$ nhood N such that $P \in N \subseteq U$

$\Longleftrightarrow$ $\forall P \in U$ $\exists$ nhood N such that $P \in N$ and $N \cap U^c = \phi$

$\Longleftrightarrow$ $\forall P \notin U^c$ ——— " ———

$\Longleftrightarrow$ $P \notin U^c \Rightarrow P \not\leftarrow U^c$

$\Longleftrightarrow$ $P \leftarrow U^c \Rightarrow P \in U^c$

$\Longleftrightarrow$ U^c is closed.

#9 Discrete topology: All sets are open so all sets are closed.

Hence One point sets and S^1 are closed.
Similarly, compact sets are closed.
Compact $\Leftrightarrow$ finite in this topology. Pf: Finite $\Rightarrow$ compact in any topology. On the other hand, an infinite set contains a sequence of distinct points, and no point will be near this.
Hence compact $\Rightarrow$ finite $\Rightarrow$ bounded in this topology.

#9 cont.

Indiscrete: One point sets are not closed since any point is near any non-empty set. (See #3)
The circle S^1 is not closed for the same reason.
Every set is compact for the same reason, and hence compact sets need not be closed or bounded.

Cofinite: A one point set is closed since its complement is a nhood.
The circle is not closed: any point is near any infinite set, since nhoods of the point must contain all but a finite number of points, so must intersect any infinite set.
Every set is compact: if a sequence takes on only finitely many values, at least one must occur infinitely often, and such a point is near the sequence.
Otherwise a sequence takes on infinitely many values, and then every point is near it because any nhood of any point contains all but finitely many points, so must intersect every tail of the sequence.
Hence, compact sets need not be closed or bounded.

Usual: One point sets are closed.
S^1 is closed.
Compact sets are closed and bounded.

#10 If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are continuous then $X \xrightarrow{gf} Z$ is also.
<u>Proof:</u> If $P \leftarrow A$ then $f(P) \leftarrow f(A)$ and so $gf(P) \leftarrow gf(A)$. //

#11 If $\mathcal{S}$ is discrete, then any function $\mathcal{S} \xrightarrow{f} X$ is continuous.
<u>Proof:</u> Let $U \subseteq X$ be open. Then $f^{-1}(U)$ is open in $\mathcal{S}$ since every set is open in the discrete topology. //

If $\mathcal{I}$ is indiscrete, then any function $X \xrightarrow{f} \mathcal{I}$ is continuous.
<u>Proof:</u> The only nhood in $\mathcal{I}$ is $\mathcal{I}$ itself, and $f^{-1}(\mathcal{I}) = X$, which is open in X. //