Solutions to Homework 4

§3 # 7, 10, 12, 13

(7) If X and Y are connected and $X \cap Y \neq \phi$ then $X \cup Y$ is connected.

Note: There are many ways to tackle this, but all seem to be based on the contrapositive: suppose $X \cup Y$ is disconnected and show that at least one of the assumptions must be false. This can be made easy by supposing all but one assumptions are true and then showing that the remaining one is false.

The key step is the Lemma proved in class:

Lemma: If C is a connected subset of X and $X = A \cup B$ is a disconnection of X then $C \subseteq A$ or $C \subseteq B$.

Proof of (7): Suppose X and Y are connected and $X \cup Y = A \cup B$ is a disconnection of $X \cup Y$. Then X and Y must each be contained in either A or B, and since both A and B are non-empty, X must be in one and Y in the other. But $A \cap B = \phi$ since no point of A is near B, and hence $X \cap Y \subseteq A \cap B$ must be empty also. //

(10) If each pair of points in S is in a connected subset of S, then S is connected.

Proof: Suppose $S = A \cup B$ with $A \neq \phi \neq B$. Let $P \in A$ and $Q \in B$. There is a connected $C \subseteq S$ with $P, Q \in C$. Since

$$C = (C \cap A) \cup (C \cap B)$$

and $P \in C \cap A$, $Q \in C \cap B$, so that $C \cap A \neq \phi$ and $C \cap B \neq \phi$, there must be a point of $C \cap A$ near $C \cap B$ or vice versa. But this gives a point of A near B or vice versa, so S is connected. //

12) was phrased carelessly in the text. Here is a clearer version.

Def: S is open if $p \in S \Rightarrow p \not\leftarrow S^c$ ($S^c = X - S$, the complement)

Prop: S is open iff each $p \in S$ has a nhood $N \subseteq S$.

Pf: ($\Rightarrow$) Suppose S open and $p \in S$. Then $p \not\leftarrow S^c$ so some nhood of p does not intersect S^c, i.e. is contained in S.
($\Leftarrow$) Suppose each $p \in S$ has a nhood contained in S. Such a nhood does not intersect S^c, so $p \not\leftarrow S^c$. //

(13) Open in $\mathbb{R}^2$: c, e, h, i
Not open : a, b, d, f, g

For reference, here is the proof of the Lemma used in (7).

Suppose C is connected and $X = A \cup B$ is a disconnection. Then

$$C = (C \cap A) \cup (C \cup B)$$

and no point of $C \cap A$ is near $C \cap B$, since no point of A is near B, and vice versa. Since C is connected, one of $C \cap A$ and $C \cap B$ is empty, so C is contained in B or in A, respectively. //