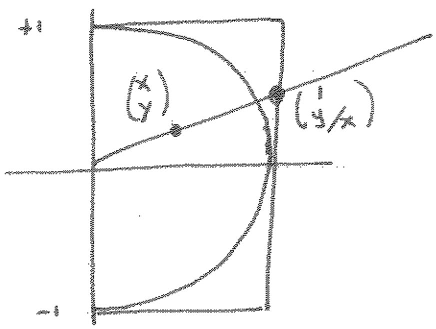
Solutions to Homework 1
M5520 W'09

1. There are many homeomorphisms $(-1,1) \to \mathbb{R}$. For example

- $f(x) = \tan\left(\frac{\pi}{2}x\right)$ and $f^{-1}(x) = \frac{2}{\pi}\tan^{-1}(x)$, or
- $f(x) = \dfrac{x}{1-x^2}$ and $f^{-1}(x) = \dfrac{2x}{1+\sqrt{1+4x^2}}$.

(It takes me about ½ page of calculations (5 large lines) to verify that this latter pair satisfies $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.)

2. The idea is to expand the disk radially so that it fills the square. Thus, the map is the identity along the x and y axes, and is multiplication by $\sqrt{2}$ along the diagonals. Clearly it is sufficient to write this out carefully in one quarter, say $x > 0$ and $|y| \le x$, and use symmetry to use the same map in the other three quarters.



Along the line through $\binom{x}{y}$ (and $\binom{1}{y/x}$) we have

distance to circle $= 1$

and

distance to square $= \sqrt{1+\frac{y^2}{x^2}} = \dfrac{\sqrt{x^2+y^2}}{x}$

Therefore Disk $\xrightarrow{\cong}$ Square should send

$$\binom{x}{y} \longmapsto \sqrt{1+\frac{y^2}{x^2}}\binom{x}{y} = \sqrt{x^2+y^2}\binom{1}{y/x}$$

and Square $\xrightarrow{\cong}$ Disk should send

$$\binom{x}{y} \longmapsto \frac{1}{\sqrt{1+\frac{y^2}{x^2}}}\binom{x}{y} = \frac{x}{\sqrt{x^2+y^2}}\binom{x}{y}$$

Standard results on rational functions and square roots show this is continuous when $x \neq 0$. More care is needed at $x = 0$. Since $|y| \le x$, if $x \to 0$ then $y \to 0$ and $\sqrt{x^2+y^2} \to 0$. Further, $0 \le \left|\frac{y}{x}\right| \le 1$, so $1 \le \sqrt{1+\frac{y^2}{x^2}} \le \sqrt{2}$ remains bounded. By the Squeeze lemma, then

$$\sqrt{1+\frac{y^2}{x^2}}\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \text{Bounded} * \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\frac{1}{\sqrt{1+\frac{y^2}{x^2}}}\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \text{Bounded} * \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so that both functions are continuous at $x=0$ as well. Finally, these functions are inverse to one another:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \sqrt{x^2+y^2}\begin{pmatrix} 1 \\ y/x \end{pmatrix} \mapsto \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+\frac{y^2}{x^2}(x^2+y^2)}}\begin{pmatrix} \sqrt{x^2+y^2} \\ \frac{y}{x}\sqrt{x^2+y^2} \end{pmatrix} = \frac{x^2+y^2}{(x^2+y^2)/x}\begin{pmatrix} 1 \\ y/x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{x}{\sqrt{x^2+y^2}}\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \sqrt{\frac{x^4+x^2y^2}{x^2+y^2}}\begin{pmatrix} 1 \\ y/x \end{pmatrix} = x\begin{pmatrix} 1 \\ y/x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

3. Since $\frac{1}{E} = \frac{1}{a} + \frac{1}{b} - \frac{1}{2}$ we first tabulate $\frac{1}{E}$:

$1/E$ (a \ b)	3	4	5	6
3	$1/6$	$1/12$	$1/30$	0
4	$1/12$	0	<0	<0
5	$1/30$	<0	<0	<0
6	0	<0	<0	<0

Now using $F = \frac{2}{a}E$ and $V = \frac{2}{b}E$

we can tabulate (V,E,F) and the type of polyhedron (omitting the impossible cases):

(V,E,F)	$b=3$	$b=4$	$b=5$
$a=3$	$(4,6,4)$ Tetrahedron	$(6,12,8)$ Octahedron	$(12,30,20)$ Icosahedron
$a=4$	$(8,12,6)$ Cube	X	X
$a=5$	$(20,30,12)$ Dodecahedron	X	X

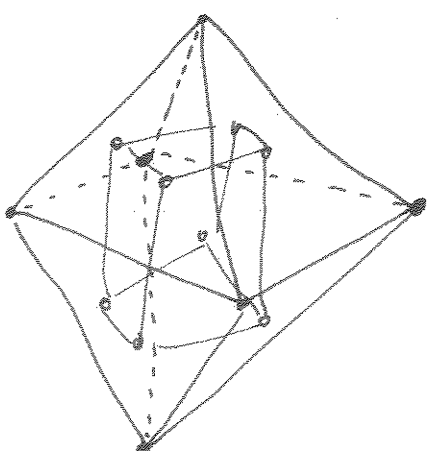
X = Impossible. No such polyhedron.

4. The tetrahedron (4,6,4) is self dual.
The cube (8,12,6) is dual to the octahedron (6,12,8).
The dodecahedron (20,30,12) is dual to the icosahedron (12,30,20).

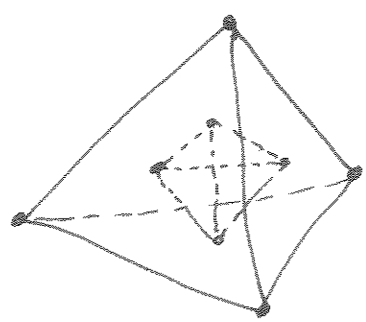
Geometrically, construct the dual as follows:

V-F) Place a vertex in the center of each face.
E-E) Connect two vertices if the faces they lie in share an edge. Thus the edges of the two polyhedra correspond.
F-V) For each vertex of the original polyhedron, the edges meeting at that vertex correspond to edges in the dual which bound a face of the dual.

Thus (V,E,F) for the dual is (F,E,V) for the original. In particular, (a,b) for the dual is (b,a) for the original.



(Good drawings are hard.)



$$\chi = V - E + F$$

5. Annulus $\chi = 8 - 12 + 4 = 0$

Double annulus $\chi = 15 - 23 + 7 = -1$

Sphere $\chi = 6 - 12 + 8 = 2$

Torus $\chi = 8 - 16 + 8 = 0$

Möbius Strip $\chi = 6 - 9 + 3 = 0$

Book $\chi = 12 - 16 + 5 = 1$

$\boxed{V - E + F = \chi}$

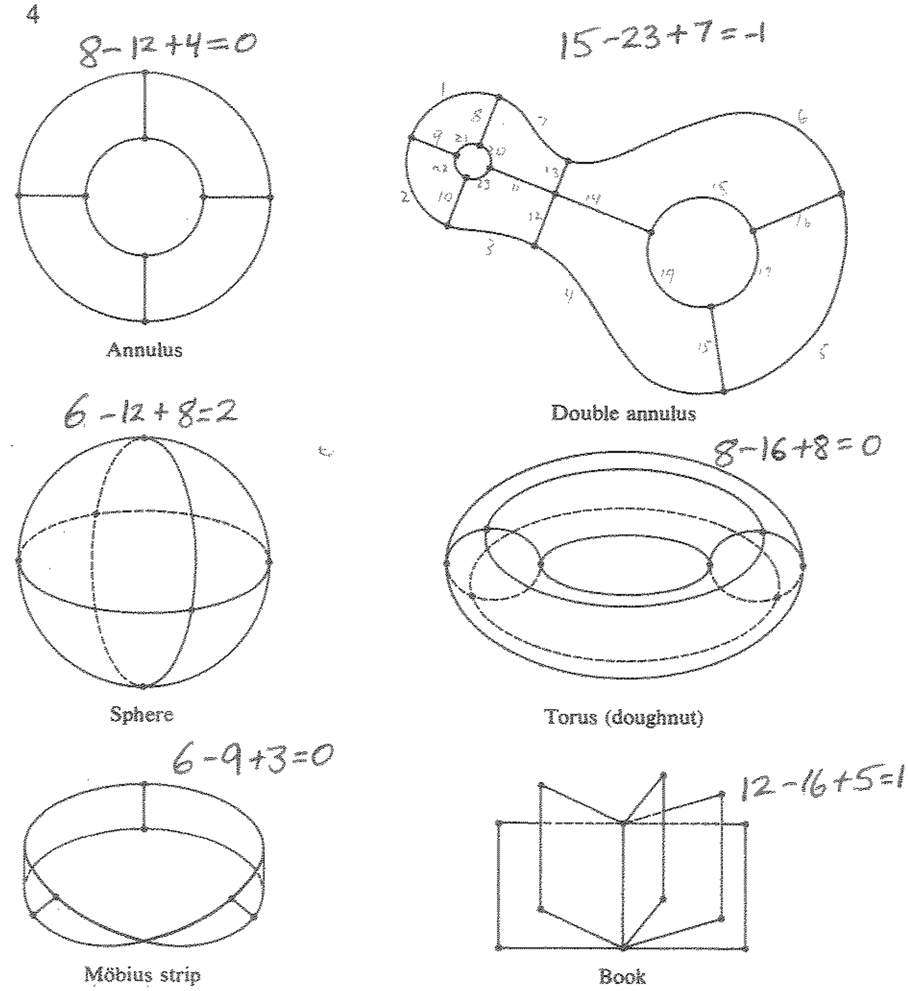


Figure 1.3 Some complexes.

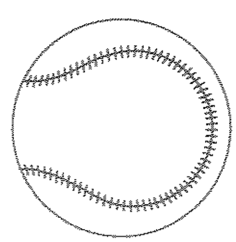


Figure 1.4 Another sphere.