MS420 F15 Test 3 Solutions

1. If $\phi : \mathbb{Z}_5 \to G$ is one-to-one then $\phi(1)$ has order 5 in G. Since $|S_4| = 24$, there are no elements of order 5 in S_4. Hence, there are no 1-1 homomorphisms $\phi : \mathbb{Z}_5 \to S_4$.

2. Ans. 1: $\phi : \mathbb{Z}_6 \to S_5$ by $\phi(1) = (123)(45)$. $(n=5)$
Ans. 2: $\phi : \mathbb{Z}_6 \to S_6$ by $\phi(1) = (1\,2\,3\,4\,5\,6)$ $(n=6)$
Note Ans 2 would give $\phi : \mathbb{Z}_6 \to S_n$ for any $n \geq 6$.
These are 1-1 homomorphisms because $\phi(1)$ has order 6.

3. $|\mathbb{Z}_2 \times \mathbb{Z}_6| = 12$, so if it were cyclic, it would contain an element of order 12. But $6(a,b) = (6a, 6b) = (0,0)$ in $\mathbb{Z}_2 \times \mathbb{Z}_6$, so the largest order of an element is ≤ 6. Hence the group is not cyclic.

4. $H = \langle (12) \rangle < S_3$ is not normal:

$$(13)H = \{(13), (13)(12)\} = \{(13), (123)\}$$

but

$$H(13) = \{(13), (12)(13)\} = \{(13), (132)\}.$$

Since $aH = Ha$ for all a, when H is normal, this shows $H = \langle (12) \rangle$ is not normal.

5. The elements of G/H are the cosets gH for $g \in G$. Every $g \in G$ is $g = a^k$ for some k, so

$$gH = a^k H = (aH)^k$$

Thus $G/H = \langle aH \rangle$, the cyclic group generated by the coset aH.

6. If $x \in aH \cap bH$ then $x = ah_1 = bh_2$ for some $h_1, h_2 \in H$. Then

$$aH = ah_1 H = bh_2 H = bH.$$

OR: $a \in aH = bH$ so $a = bk$ for some k in H. Thus $aH = bkH = bH$. //

7. (a)

$$\begin{array}{r|l} & x^3 \\ \hline x^4 - 1 & x^7 \quad\quad -1 \\ & x^7 - x^3 \\ \hline & \quad x^3 - 1 \end{array}$$

$$\begin{array}{r|l} & x \\ \hline x^3 - 1 & x^4 \quad\quad -1 \\ & x^3 - x \\ \hline & \quad x - 1 \end{array}$$

and

$$x^3 - 1 = (x-1)(x^2+x+1)$$

So

$$\boxed{\mathrm{Gcd}(x^4-1, x^7-1) = x - 1}$$

(b) So

$$x^7 - 1 = x^3(x^4-1) + (x^3-1)$$
$$x^4 - 1 = x(x^3-1) + (x-1)$$

Then

$$(x-1) = (x^4-1) - x(x^3-1)$$
$$(x-1) = (x^4-1) - x\left[(x^7-1) - x^3(x^4-1)\right]$$
$$(x-1) = (x^4+1)(x^4-1) - x(x^7-1)$$

(b) $$\boxed{x-1 = (x^4+1)(x^4-1) - x(x^7-1)}$$

~~8. $x^{100} + x^{99} - 1 = (x-2)g(x) + f(2)$, where $f(x) = x^{100} + x^{99} - 1$~~

8. Let $f(x) = x^{100} + x^{99} - 1 \in \mathbb{Z}_3[x]$. By the remainder theorem

$$f(x) = (x-2)g(x) + f(2),$$

that is, the remainder will be $f(2) = 2^{100} + 2^{99} - 1$. Now in $\mathbb{Z}_3$, $2^2 = 1$. Hence $2^{100} = 1$ and $2^{99} = 2$. Thus, the remainder is

$$2^{100} + 2^{99} - 1 = 1 + 2 - 1 = 2 = -1 .$$

9. $$f(x) \neq 0 \text{ and } f(x)g(x) = f(x)h(x) \implies g(x) = h(x)$$

is equivalent to

$f(x) \neq 0$ and $f(x)g(x) = 0 \implies g(x) = 0$, so we prove this. Let $f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$ with $a_m \neq 0$ and $g(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$ with $b_n \neq 0$. Then $f(x)g(x) = a_m b_n x^{n+m} + (a_m b_{n-1} + a_{m-1} b_n)x^{n+m-1} + \cdots + a_0 b_0$ and $a_m b_n \neq 0$, so $f(x)g(x) \neq 0$. Thus,

$$f(x) \neq 0 \text{ and } g(x) \neq 0 \implies f(x)g(x) \neq 0$$

or

$$f(x) \neq 0 \text{ and } f(x)g(x) = 0 \implies g(x) = 0. //$$

10. If $c \in \mathbb{Z}_5$ then $c^5 = c$ by Fermat's little theorem. Thus

$$x^5 - x = x(x-1)(x-2)(x-3)(x-4).$$