M5420 F15 TEST 2 SOLUTIONS

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1. (a) $(123)(234) = (12)(34)$ order 2

(b) $(123)(234)(345) = (12)(34)(345) = (12)(45)$ order 2

(c) $(1234)(2345) = (12453)$ order 5

(d) $(1234)(3456) = (123)(456)$ order 3

(e) $(1234)(4567) = (1234567)$ order 7

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2. $\mathbb{Z}_6$, $0 = \{0\}$, $2\mathbb{Z}_6 = \{0,2,4\}$ and $3\mathbb{Z}_6 = \{0,3\}$.

If a subgp contains 1 or $5 = -1$, it is $\mathbb{Z}_6$.

If it contains 2, or 4, it contains $\{0,2,4\}$

If it contains 3, it contains $\{0,3\}$. So this is the whole list.

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3. $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{\phi} \mathbb{Z}_8^\times$ by $\phi(n,m) = 3^n 5^m$ is an isomorphism.

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4. $|G| = p$, prime. $|\langle a\rangle| \neq 1$ since $a \neq e$ and $|\langle a\rangle| \mid p$, so $|\langle a\rangle| = p$.
Thus $\langle a\rangle$ contains all of G, i.e. $\langle a\rangle = G$.

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5. Let $n = |G|$. We are given that n is odd, so $1 = an + 2b$ for some integers a and b. Then $g = g^1 = g^{an+2b} = (g^n)^a (g^2)^b = ee = e$, since $g^n = e$ by the Corollary to Lagrange's Thm and we are given that $g^2 = e$.

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6. (a) $|S_3| = 6 \neq |S_4| = 24$ so there is no 1-1, onto function between them

(b) S_3 is not abelian, while $\mathbb{Z}_6$ is abelian, so there is no isomorphism between them.

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7. (a) If $g^2 = e$ then $g^2 g^{-1} = e g^{-1}$, or $g = g^{-1}$.
If $g = g^{-1}$ then $g^2 = g g^{-1} = e$.

(b) Consider the equivalence relation $g \sim g^{-1}$ on G.
$[e] = \{e\}$ has only one element, ~~so the~~
The equivalence classes are

$$[g] = \begin{cases} \{g\} & g = g^{-1}, \text{ i.e. } g^2 = e \\ \{g, g^{-1}\} & g \neq g^{-1} \end{cases}$$

Since $G \setminus \{e\}$ has an odd number of elements, some equivalence class has size 1. This class is $\{g\}$, where $g^2 = e$.

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8. Let $I = \{k \in \mathbb{Z} \mid g^k = e\}$ where G is a group & $g \in G$.

(a) If $j, k \in I$ then $g^j = g^k = e$ so $g^{j+k} = g^j g^k = ee = e$. Hence $j+k \in I$. Similarly, $g^{j-k} = g^j (g^k)^{-1} = ee^{-1} = e$, so $j-k \in I$.

(b) If $d \neq 0$ it is the <u>order of g</u>.
If $d = 0$ we say <u>g has order ∞</u> and d isn't anything in particular.