Math 5420, Fall 2014, Take Home Test 4
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Due at 8:00 AM on Monday, 15 December, at the start of the final exam in 228 State Hall (our regular classroom).

Your work should all be your own. You are free to discuss these problems with me, and to discuss general mathematical issues with anyone.

1. An *idempotent* e in a ring is an element such that $e^2 = e$. Show that if e is an idempotent then $1 - e$ is also an idempotent.

2. Find all the idempotents in $\mathbf{Z}[x]/\langle x^2 - 1\rangle$ and $\mathbf{Q}[x]/\langle x^2 - 1\rangle$, and group them into pairs of the form e, $1 - e$.

3. (a) Let p be a prime. How many elements of order p are there in the group $\mathbf{Z}_p \times \mathbf{Z}_p$?

   (b) If F is a field, show that the multiplicative group $F^\times$ cannot have a subgroup isomorphic to $\mathbf{Z}_p \times \mathbf{Z}_p$. (Hint: how many solutions can the polynomial $x^p - 1$ have?)

4. If $\zeta = e^{2\pi i/7}$ then $2\cos(2\pi/7) = (\zeta + \zeta^6)$. Find a cubic polynomial satisfied by $\zeta + \zeta^6$.

In the remaining problems, let D be an integral domain. Define a relation on the set

$$D \times D^\times = \{(a, b) \in D^2 \mid b \neq 0\}$$

by setting $(a, b) \sim (c, d)$ if and only if $ad = bc$.

5. Show that this is an equivalence relation.

———— Continued on reverse ————

Now define F to be the set of equivalence classes,

$$F = (D \times D^{\times})/\!\sim \ = \{[a, b] \mid b \neq 0\}$$

where $[a, b]$ denotes the equivalence class of (a, b).

Define addition and multiplication on F by

$$[a, b] + [c, d] = [ad + bc, bd]$$

and

$$[a, b][c, d] = [ac, bd].$$

6. Show that addition and multiplication are well defined.
7. Show that addition and multiplication are associative.
8. Show that addition and multiplication are commutative.
9. Show that addition and multiplication have identity elements, '0' and '1', and identify these elements.
10. Show that any element $[a, b] \in F$ has an additive inverse.
11. Show that any nonzero element $[a, b] \in F$ has a multiplicative inverse.
12. Show that the function $i : D \longrightarrow F$ given by $i(a) = [a, 1]$ is a homomorphism and that it is one-to-one.

——————— The End ———————