Math 5420, Fall 2014, Test 3
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Read all the problems quickly before starting work.
Turn in your bluebook and keep this list of questions for later reference.

1. (10 points) Is $\langle(1234)\rangle$ a normal subgroup of S_4?

2. (10 points) Define one-to-one homomorphisms $\mathbf{Z}_4 \longrightarrow S_4$ and $\mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow S_4$.

3. (10 points) Let $H < G$ and $a, b \in G$. Show that if $aH = Hb$ then $aH = Ha$.

4. (10 points) Recall that the center of a group G is

$$Z(G) = \{g \in G \mid ag = ga \text{ for all } a \in G\}.$$

   Show that the $Z(G)$ is a normal subgroup.

5. (10 points) Let $f : G \longrightarrow H$ be a group homomorphism. Show that $K = \ker(f)$ is a normal subgroup of G.

6. (10 points) Show that in a field F, $a \cdot 0 = 0$ for all $a \in F$.
   (Use only the definition of a field; that is, do not just say 'Proposition X says this is true'.)

7. Let F be a field.

   (a) (10 points) Show that if $a, b \in F$ and $ab = 0$, then $a = 0$ or $b = 0$.

   (b) (5 points) Show that if $f, g \in F[x]$ and $fg = 0$, then $f = 0$ or $g = 0$.

8. (a) (8 points) In $\mathbf{Q}[x]$, compute $\gcd(x^3 - 1, x^5 - 1)$.

   (b) (7 points) Write the gcd as a linear combination of $x^3 - 1$ and $x^5 - 1$.

9. (10 points) Factor $x^4 + 3x^2 + 2$ into irreducible factors in $\mathbf{Z}_5[x]$ and find its roots in $\mathbf{Z}_5$.

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