Math 5420, Fall 2014, Test 2
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Read all the problems quickly before starting work.
Turn in your bluebook only.
Keep this list of questions for later reference.

1. Define an isomorphism $\phi : \mathbf{Z}_6 \longrightarrow \mathbf{Z}_7^\times$.

2. Write the permutation $(345)(1234)(123)$ as a product of disjoint cycles and compute its order.

3. Compute all powers of the permutation (123456), writing them as disjoint cycles and computing their orders.

4. Let H be a subgroup of G with $|G| = 7|H|$. If K is a subgroup of G which contains H, $H < K < G$, show that either $K = H$ or $K = G$.

5. Let G be a group and let $H < G$ be a subgroup.

   (a) Define a relation on G by

   $$x \sim y \iff xy \in H$$

   Is this an equivalence relation? Prove or disprove.

   (b) Define a relation on G by

   $$x \sim y \iff xy^{-1} \in H$$

   Is this an equivalence relation? Prove or disprove.

6. Prove that if G is a finite group of order N, then $x^N = e$ for every $x \in G$. (Hint: recall that the order of an element x is the order of the subgroup $\langle x \rangle$ that it generates.)

——————— Continued on reverse ———————

7. (a) Let $N = \text{lcm}(n, m)$. Show that $N(x, y) = (0, 0)$ for every $(x, y) \in \mathbf{Z}_n \times \mathbf{Z}_m$.

   (b) If $gcd(n, m) > 1$ show that $\mathbf{Z}_n \times \mathbf{Z}_m$ is not cyclic.

8. Let G be a group. Define a relation on G by $g \sim h$ iff $xgx^{-1} = h$ for some $x \in G$.

   (a) Show that this is an equivalence relation.

   (b) Divide S_3 into equivalence classes. Hint: recall that $\sigma(1, 2, 3, \ldots, k)\sigma^{-1} = (\sigma(1), \sigma(2), \ldots, \sigma(k))$.

9. Let G be a group and let $g \in G$.

   (a) Let $f : G \longrightarrow G$ be $f(x) = gx$. Show that f is one-to-one and onto. Is f an isomorphism? Prove your answers.

   (b) Let $h : G \longrightarrow G$ be $h(x) = gxg^{-1}$. Show that h is one-to-one and onto. Is h an isomorphism? Prove your answers.

——————— The End ———————