Test 3 Solutions

1. Since $(12)(1234)(12)^{-1} = (2134)$, and this is not in $\langle(1234)\rangle$, the subgroup $\langle(1234)\rangle$ is not normal.

2. $\mathbb{Z}_4 \xrightarrow{\phi} S_4$ by $\phi(k) = (1234)^k$ is one-to-one since $o((1234)) = 4$.

   $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{\psi} S_4$ by $\phi(n,m) = (12)^n(34)^m$ is a homomorphism because (12) and (34) are commuting elements of order 2, and is one-to-one.

3. Suppose $aH = Hb$. Then $a = ae \in aH = Hb$, and so $a = hb$ for some $h \in H$. Then $Ha = Hhb = Hb$, since $Hh = H$.

4. Suppose $g \in Z(G)$, and $x \in G$. Then $xg = gx$, so $xgx^{-1} = gxx^{-1} = g$, which is in $Z(G)$. Hence $Z(G)$ is normal.

5. Suppose $k \in Ker(f)$. Then $f(k) = e$. Therefore $f(gkg^{-1}) = f(g)ef(g)^{-1} = e$, so $gkg^{-1} \in Ker(f)$ as well. Thus, $Ker(f)$ is normal.

6. $a \cdot 0 = a \cdot (0 + 0)$ since $x = x + 0$ for any x

   $= a \cdot 0 + a \cdot 0$ by distributive law.

   Subtracting $a \cdot 0$ from each side, we get $0 = a \cdot 0$.

7. (a) Suppose $ab = 0$. If $a = 0$, we are done. If not, $a^{-1} \in F$, and we get $a^{-1}(ab) = a^{-1}(0) = 0$, or $b = 0$.

   (b) Write $f(x) = a_n x^n +$ lower degree terms and $g(x) = b_m x^m +$ lower. Then $f(x)g(x) = a_n b_m x^{n+m}$ + lower degree terms. If a_n and b_m are non-zero, so is $a_n b_m$, by part (a).

8.

8.

(a)

$$\begin{array}{r|l} & x^2 \\ \hline x^3-1 & x^5 \qquad -1 \\ & \underline{x^3-x^2} \\ & \quad x^2-1 \end{array} \qquad \begin{array}{r|l} & x \\ \hline x^2-1 & x^3 \qquad -1 \\ & \underline{x^3-x} \\ & \quad x-1 \end{array} \qquad \begin{array}{r|l} & x+1 \\ \hline x-1 & x^2 \qquad -1 \end{array}$$

$gcd = x \quad 1$

(b)

$$\begin{aligned} x-1 &= (x^3-1) - x(x^2-1) \\ &= (x^3-1) - x\left((x^5-1) - x^2(x^3-1)\right) \\ &= (x^3+1)\cdot(x^3-1) - x\cdot(x^5-1) \end{aligned}$$

9. $(x^4+3x^2+2) = (x^2+1)(x^2+2)$ in any $F[x]$.

$x^2+1 = x^2-4 = (x-2)(x+2)$ in $\mathbb{Z}_5[x]$

To see that x^2+2 does not factor in $\mathbb{Z}_5[x]$, it is sufficient to observe that it has no roots: -2 is not a square mod 5:

x	0	1	2	3	4
x^2	0	1	4	4	1

Hence, the complete factorization of x^4+3x^2+2 in $\mathbb{Z}_5[x]$ is

$$(x-2)(x+2)(x^2+2).$$