USE YOUR IMAGINATION™

Blue Book

EXAMINATION BOOK

BOX NO. ____________

NAME Key Test 2

SUBJECT m5420 F14

CLASS

SECTION

INSTRUCTOR

DATE

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1. Define an iso $\phi : \mathbb{Z}_6 \to \mathbb{Z}_7^{\times}$.

Note that in $\mathbb{Z}_7^{\times}$ we have $2^2 = 4, 2^3 = 8 = 1$, so $[2]_7$ does not generate.

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$3^2 = 9 = 2$, so $3^3 = 3 \cdot 2 = 6 = -1 \neq 1$, so $o(3) \neq 1, 2,$ or 3, hence

(3 1-1, onto / 7 hom)

3 generates $\mathbb{Z}_7^{\times}$. Then $\phi : \mathbb{Z}_6 \to \mathbb{Z}_7^{\times}$ by $\phi(k) = [3]^k$ is an iso.

6, 4

2. $(345)(1234)(123) = (14)(253)$ so the order is 6.

6, 4

3.

$(123456)^0 = ()$ order $= 1$

$(\quad)^1 = (123456)$ 6

$(\quad)^2 = (135)(246)$ 3

$(\quad)^3 = (14)(25)(36)$ 2

$(\quad)^4 = (153)(264)$ 3

$(\quad)^5 = (165432)$ 6

10

4. $H < K < G \implies |H| \mid |K| \mid |G| = 7|H|$. Thus $|K| = a|H|$ for some a, and $a \mid 7$. Thus $a = 1$ or 7. If $a = 1$ then $H = K$, while if $a = 7$, $K = G$.

$|K| \mid 7|G| \not\Rightarrow |K| \mid 7$ or $|K| \mid |G|$

unless $|K|$ is prime.

In fact $|K| \mid |G|$ since $K < G$.

5.

5. $H < G$

(a) $x \sim y \iff xy \in H$

Is $\sim$ reflexive? $x \sim x \iff x^2 \in H$ and this is not true in general,

4

So we have no reason to believe that $\sim$ is an equivalence relation. That is, for some $H < G$, it is not.

6:

(b) $x \sim y \iff xy^{-1} \in H$

2 Reflexive: $x \sim x \iff xx^{-1} \in H \iff e \in H$ and this is true.

2 Symmetric: $x \sim y \iff xy^{-1} \in H$

$\iff yx^{-1} \in H$ since H is closed under $()^{-1}$

$\iff y \sim x$ so $\sim$ is symmetric

2 Transitive $x \sim y$ and $y \sim x \implies xy^{-1} \in H$ and $yz^{-1} \in H$

$\implies xy^{-1}yz^{-1} \in H$ since H is closed under products

$\implies xz^{-1} \in H$

$\implies x \sim z$ so $\sim$ is Transitive.

$\sim$ is an equivalence relation.

6. $o(x) = |\langle x \rangle|$ and this divides N, the order of G, by Lagrange's theorem. Say $N = k\, o(x)$. Then

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$$x^N = x^{k \cdot o(x)} = \left(x^{o(x)}\right)^k = e^k = e.$$

7. $\mathbb{Z}_{28}$ has an element of order 28, but $14(x,y) = 0$ for every $(x,y) \in \mathbb{Z}_2 \times \mathbb{Z}_{14}$. Therefore they cannot be isomorphic.

8. (a) $g \sim h \iff xgx^{-1} = h$ for some $x \in G$

This is an equivalence relation:

2 <u>Reflexive:</u> let $x = e$: $ege^{-1} = g$ so $g \sim g$

2 <u>Symmetric:</u> if $g \sim h$ then $xgx^{-1} = h$ for some $x \in G$

so $g = x^{-1}hx$ (and $x^{-1} \in G$)

so $h \sim g$.

2 <u>Transitive:</u> if $g \sim h$ and $h \sim k$ then

$xgx^{-1} = h$ and $yhy^{-1} = k$ for some $x, y \in G$.

$=6$

Then $\cancel{xyhy^{-1}x^{-1} = xk}$

$yxgx^{-1}y^{-1} = yhy^{-1} = k$

or

$(yx)g(yx)^{-1} = k$ (and $yx \in G$)

so $g \sim k$.

(b) $\sigma()\sigma^{-1} = ()$ so $[()] = \{()\}$

$\sigma(1\,2)\sigma^{-1} = (\sigma(1)\ \sigma(2))$ so $[(1\,2)] = \{(1\,2), (1\,3), (2\,3)\}$

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using $\sigma = e$, $\sigma = (2\,3)$, $\sigma = (1\,2\,3)$ resp.

$\sigma(1\,2\,3)\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3))$ so $[(1\,2\,3)] = \{(1\,2\,3), (1\,3\,2)\}$

using $\sigma = e$ and $\sigma = (2\,3)$ resp.

Three equivalence classes.

7 (a) $N = an = bm$ so

5 $N(x,y) = (anx, bmy) = (a\cdot 0, b\cdot 0) = (0,0)$

5 (b) $\mathbb{Z}_n \times \mathbb{Z}_m$ cyclic $\iff \mathbb{Z}_n \times \mathbb{Z}_m = \langle (x,y) \rangle$ for some (x,y).

But $|\langle (x,y) \rangle| < |\mathbb{Z}_n \times \mathbb{Z}_m|$ by part (a), if $\gcd(n,m) > 1$, since $|\langle (x,y) \rangle| \le N = km < mm$, so $\langle (x,y) \rangle \neq \mathbb{Z}_n \times \mathbb{Z}_m$ for every (x,y), when $\gcd(n,m) > 1$.

9. G is a group and $g \in G$

(a) $f: G \to G$ is $f(x) = gx$.

3 f is one-to-one: if $gx = gy$ then $g^{-1}gx = g^{-1}gy$, so $x = y$.

3 f is onto: if $h \in G$ then $g(g^{-1}h) = h$

4 Thus f is an isomorphism $\iff$ f is a homomorphism.

Now,

10 $$f(xy) = gxy \quad \text{while } f(x)f(y) = gxgy$$

We have

$$\begin{aligned} & gxy = gxgy \\ \iff \ & x^{-1}g^{-1}(gxy)y^{-1} = x^{-1}g^{-1}gxgyy^{-1} \\ \iff \ & yy^{-1} = gyy^{-1} \\ \iff \ & g = e \end{aligned}$$

so, no, in general f is not an isomorphism. only when $g = e$ & $f =$ identity.

(b) $G \xrightarrow{h} G$ by $h(x) = gxg^{-1}$

3 one-to-one: $gxg^{-1} = gyg^{-1} \Rightarrow x = y$ by $g^{-1}(\ )g$ of both side

3 onto: $gxg^{-1} = y \Rightarrow x = g^{-1}yg$ so $h(g^{-1}yg) = y$

4 hom: $h(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = h(x)h(y)$. //

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