§3.8 #5, 6, 11, 12, 13

5) If $H < G$ show there is a 1-1 correspondence between G/H and $H\backslash G$.

Proof: Let $\alpha: G/H \to H\backslash G$ be $\alpha(gH) = Hg^{-1}$ and
$\beta: H\backslash G \to G/H$ be $\beta(Hg) = g^{-1}H$.
These are well defined functions:
if $g_1H = g_2H$ then $g_1 = g_2h$ for some $h \in H$. Then $Hg_1^{-1} = Hh^{-1}g_2^{-1} = Hg_2^{-1}$, so $\alpha(g_1H) = \alpha(g_2H)$. That is α is welldefined.
Similarly β is well defined.
They are mutually inverse $\alpha\beta(Hg) = \alpha(g^{-1}H) = Hg$ and $\beta\alpha(gH) = \beta(Hg^{-1}) = gH$. / Hence they are 1-1 and onto. //

6) If $H < G$ and $N \triangleleft G$ then $H \cap N \triangleleft H$.

Proof: We have already shown that $H \cap N$ is a subgroup of G, and hence of H. If $h \in H$ and $n \in H \cap N$ then $hnh^{-1} \in hNh^{-1} = N$. Since $h, n \in H$, $hnh^{-1} \in H$ as well; so $hnh^{-1} \in H \cap N$. //

11) If $N \triangleleft G$ then $o(aN)$ in G/N divides $o(a)$, when $o(a) < \infty$.

Proof: $(aN)^{o(a)} = a^{o(a)}N = eN = N$ so $o(aN) \mid o(a)$. //

12) Let $H \triangleleft G$ and $K \triangleleft G$ and $H \cap K = \{e\}$. Show that $hk = kh$ for all $h \in H$ and $k \in K$.

Proof: Note that $hk = kh$ iff $hkh^{-1}k^{-1} = e$. Now $hkh^{-1}k^{-1} = (hkh^{-1})k^{-1} \in K$ since $K \triangleleft G$ implies $hkh^{-1} \in K$, and $hkh^{-1}k^{-1} = h(khk^{-1}) \in H$ since $khk^{-1} \in H$ since $H \triangleleft G$. Thus

$$hkh^{-1}k^{-1} \in H \cap K$$

and hence $hkh^{-1}k^{-1} = e$. //

13) Let $N \triangleleft G$. Show G/N is abelian iff $N \supseteq \{aba^{-1}b^{-1} \mid a, b \in G\}$.

Proof: Suppose G/N abelian. Then $aNbN = abN$ and $bNaN = baN$ are the same, so $a^{-1}b^{-1}ab \in N$ for every $a, b \in G$. Conversely, if so then $aNbN = abN = ab(b^{-1}a^{-1}ba)N = baN$ since $b^{-1}a^{-1}ab \in N$. Hence G/N is abelian. //