§3.6 # 11, 25

§3.7 # 7, 8, 13

§3.6

(11) Show D_n is isomorphic to a subgroup of S_n.

Proof: Using the presentation $D_n = \langle a, b \mid a^n = e = b^2, ba = a^{-1}b \rangle$ we can define a 1-1 homomorphism $D_n \xrightarrow{\phi} S_n$ by

$$\phi(a) = (1\ 2 \cdots n), \quad \phi(b) = (2, n)(3, n-1)(4, n-2)\cdots$$

Certainly, $\phi(a)^n = (1\ 2 \cdots n)^n = ()$, $\phi(b)^2 = ()$ and

$$\phi(b)\phi(a)\phi(b)^{-1} = (2\ n)(3\ n-1)\cdots(1\ 2\ 3 \cdots n)(2\ n)(3\ n-1)\cdots$$

$$= (1\ n\ n-1\ \cdots\ 2) = \phi(a)^{-1}$$

so ϕ is a homomorphism. To show it is 1-1 we will compute $\ker\phi$. Certainly, $\phi(a^i)$ sends 1 to $i+1 \pmod n$ so $\phi(a)^i = e$ only if $i \equiv 0\ (n)$, in which case $a^i = e$. Similarly, since $\phi(b)$ leaves 1 fixed, $\phi(a^i b)$ also sends 1 to $i+1$, so $\phi(a^i b) = e$ would require $a^i = e$, as before. But $\phi(b) \neq e$ either, so $\ker\phi = \{e\}$. //

Alternative conceptual proof. Let us number the vertices of the regular n-gon by $1, 2, 3, \ldots, n$. Then, any symmetry of the n-gon must send vertices to vertices, hence defines a function $D_n \to S_n$. This is 1-1 because two symmetries which have the same effect on vertices must be identical. It is a homomorphism because in both groups the operation is composition, and this function is simply restriction of the symmetry from the whole n-gon to its set of vertices. //

(25) Define an isomorphism from S_n to a subgroup of A_{n+2}.

Definition and proof: For $\sigma \in S_n$ let $\mathrm{sign}(\sigma) = 0$ if σ is even and $\mathrm{sign}(\sigma) = 1$ if σ is odd. We showed earlier that sign is a homomorphism $S_n \longrightarrow \mathbb{Z}_2$; that is $\mathrm{sign}(\sigma\tau) = \mathrm{sign}(\sigma) + \mathrm{sign}(\tau)$. Define $\phi: S_n \to A_{n+2}$ by $\phi(\sigma) = \sigma\,(n+1\ n+2)^{\mathrm{sign}(\sigma)}$. Then

$$\phi(\sigma\tau) = \sigma\tau\,(n+1\ n+2)^{\mathrm{sig}(\sigma\tau)} = \sigma\tau\,(n+1\ n+2)^{\mathrm{sign}(\sigma)}(n+1\ n+2)^{\mathrm{sign}(\tau)}$$

$$= \sigma\,(n+1\ n+2)^{\mathrm{sign}(\sigma)}\,\tau\,(n+1\ n+2)^{\mathrm{sign}(\tau)} = \phi(\sigma)\phi(\tau)$$

$\mathrm{sign}(\sigma)$

does not move any of $1, 2, \ldots, n$, and hence $\sigma = e$. Thus ϕ is 1-1 and $S_n \cong \phi(S_n)$ is a subgroup of A_{n+2}. //

§3.7

(7) (a) hom: $\phi(ab) = \begin{bmatrix} ab & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} = \phi(a)\phi(b)$

(b) hom: $\phi(a)\phi(b) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \phi(a+b)$

(c) hom: $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = \phi\begin{bmatrix} a+a' & \cdot \\ - & \cdot \end{bmatrix} = a + a' = \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + \phi\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right)$

(d) not: $\phi\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0 \notin \mathbb{R}^\times$ (ϕ also fails to preserve products)

(e) not: $\phi\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2$ is not $e \in \mathbb{R}$ (—— " ——)

(f) hom: $\det\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \det\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = a_1a_2d_1d_2 - a_1b_2c_2d_1 - a_2b_1c_1d_2 + b_1b_2c_1c_2$

while $\det\begin{bmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2 \\ c_1a_2+d_1c_2 & c_1b_2+d_1d_2 \end{bmatrix} = a_1a_2d_1d_2 + b_1b_2c_1c_2 - a_2b_1c_1d_2 - a_1b_2c_2d_1$.

(8) If $G_1 \xrightarrow{\phi} G_2$ and $G_2 \xrightarrow{\theta} G_3$ are group homomorphisms then $\theta\phi$ is a group homomorphism and $\operatorname{Ker}\phi \subseteq \operatorname{Ker}\theta\phi$.

Proof: $(\theta\phi)(ab) = \theta(\phi(ab)) = \theta(\phi(a)\phi(b)) = \theta(\phi(a))\,\theta(\phi(b))$ $= (\theta\phi)(a)\,(\theta\phi)(b)$, so $\theta\phi$ is a group homomorphism.

If $\phi(a) = e$ then $\theta\phi(a) = \theta(e) = e$, so $\ker(\phi) \subseteq \ker\theta\phi$. //

(13) If $H < G$ then $H \triangleleft G$ iff $\forall a \in G, h \in H$ $\exists h' \in H$ s.t. $ah = h'a$.

Proof: If $H \triangleleft G$ then $aha^{-1} \in H$, so $aha^{-1} = h' \in H$, or $ah = h'a$.

Conversely, if $\exists h'$ s.t. $ah = h'a$ for $a \in G, h \in H$, then $aha^{-1} = h'aa^{-1}$ $= h' \in H$ and so $aHa^{-1} \subseteq H$ for each $a \in G$. That is, $H \triangleleft G$. //