§3.2 # 6, 11, 17, 19, 21

(6) Let $G = Gl_2(\mathbb{R})$ and $T = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| ad \neq 0 \right\}$ and $D = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \middle| ad \neq 0 \right\}$.

(a) T is a subgroup of G

Proof: Each $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ in T is in G since $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & -b/ad \\ 0 & 1/d \end{bmatrix}$ exists, because $ad \neq 0$. The inverse is in T as well, so T is closed under inverses. For closure under products, note that if $a_1 d_1 \neq 0$ and $a_2 d_2 \neq 0$ then

$$\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & a_1 b_2 + b_1 d_2 \\ 0 & d_1 d_2 \end{bmatrix} \in T$$

since $a_1 a_2 d_1 d_2 \neq 0$. //

(b) D is a subgroup of G (In fact, of T).

Proof: $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/d \end{bmatrix} \in D$ since $ad \neq 0$. Similarly,

$$\begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & 0 \\ 0 & d_1 d_2 \end{bmatrix} \in D \text{ as well.} //$$

(11) Let S be a set, $a \in S$, and $G = \{\sigma \in Sym(S) \mid \sigma(a) = a\}$. Show G is a subgroup of $Sym(S)$.

Proof: If $\sigma \in G$ then $\sigma(a) = a$ so $\sigma^{-1}(\sigma(a)) = \sigma^{-1}(a)$, or $a = \sigma^{-1}(a)$. Hence $\sigma^{-1} \in G$ also. If $\sigma, \tau \in G$ then $(\sigma\tau)(a) = \sigma(\tau(a)) = \sigma(a) = a$, so $\sigma\tau \in G$ as well. //

(17) Let $\{G_\alpha \mid \alpha \in A\}$ be a collection of subgroups of G. Then $\bigcap_{\alpha \in A} G_\alpha$ is a subgroup of G.

Proof: If $g \in \bigcap_\alpha G_\alpha$ then $g \in G_\alpha$ for each α, so $g^{-1} \in G_\alpha$ for each α, so $g^{-1} \in \bigcap_\alpha G_\alpha$. Similarly, if $g, h \in \bigcap_\alpha G_\alpha$ then $g, h \in G_\alpha$ for each α,

(17) (cont.)
so $gh \in G_\alpha$ for each α and hence $gh \in \bigcap_\alpha G_\alpha$. //

(19) Let G be a group and $a \in G$. The set $C_G(a)$ or $C(a) = \{x \in G \mid xa = ax\}$ is the <u>centralizer of a in G</u>.

(a) $C(a)$ is a subgroup of G.
(b) $\langle a \rangle \leq C(a)$.
(c) Compute $C_{S_3}((123))$.
(d) Compute $C_{S_3}((12))$.

<u>Proofs</u>: (a) If $x, y \in C(a)$ then $(xy)a = x(ya) = x(ay) = (ax)y = a(xy)$ so $xy \in C(a)$. Also $x^{-1}a = x^{-1}axx^{-1} = x^{-1}xax^{-1} = ax^{-1}$, so $x^{-1} \in C(a)$. Hence $C(a)$ is a subgroup.
(b) $a^n a = a^{n+1} = aa^n$ so $a^n \in C(a)$ for any $n \in \mathbb{Z}$. Thus $\langle a \rangle \subset C(a)$. // <u>Alternative proof:</u> use the

<u>Lemma</u>: If S is a subset of G, H a subgroup of G and $S \subseteq H$ then $\langle S \rangle \leq H$.
<u>Pf</u>: H is closed under products and inverses, so contains all words in S and S^{-1}. //

Then you need only show $a \in C(a)$ to conclude $\langle a \rangle \leq C(a)$, and this is easy: $aa = aa$, hence $a \in C(a)$. //
(c) Since $\langle (123) \rangle \leq C((123))$ and $o(\langle (123) \rangle) = 3$, $C((123))$ is either $\langle (123) \rangle$ or all of S_3, because

$$3 = o\langle (123) \rangle \leq o(C((123))) \leq o(S_3) = 6.$$

But $(12)(123) = (23)$ while $(123)(12) = (13)$, so $C((123)) = \langle (123) \rangle$.
(d) $o(\langle (12) \rangle) = 2 \leq o(C((12))) \leq o(S_3) = 6$ so ETS $(123) \notin C((12))$ to see that $C(\langle (12) \rangle) = \langle (12) \rangle$. //

(21)

21 $Z(G) = \{x \in G \mid \forall g \in G \quad xg = gx\}$

(a) Show $Z(G)$ is a subgroup.

(b) Show $Z(G) = \bigcap_{a \in G} C(a)$

(c) $Z(S_3)$

Proof (a) Either apply (b) and the homework from last time, or; add $\forall a \in G$ to the proof of Exer 19(a).

(b) $x \in Z(G)$ implies $xg = gx$ for all $g \in G$, so $x \in C(g)$ for all $g \in G$. Hence $x \in \bigcap_g C(g)$.

(c)
$$Z(S_3) \subseteq Z(\langle(12)\rangle) \cap C(\langle(123)\rangle) = \langle(12)\rangle \cap \langle(123)\rangle = e. //$$