Section 3.1

(10) Show that $A = \{f_{m,b} : \mathbb{R} \to \mathbb{R} \mid m, b \in \mathbb{R},\ m \neq 0,\ f_{m,b}(x) = mx+b\}$ is a group under composition.

Proof: Composition is a well defined operation on functions which is associative and has identity operation the function $id(x) = x$. Since $f_{1,0}(x) = 1x + 0 = x$, $f_{1,0} = id$ is in A.

Closure follows: from $f_{m,b}(f_{n,c}(x)) = f_{mn, mc+b}(x)$, so $f_{m,b} \circ f_{n,c} = f_{nm, mc+b} \in A$.

Each element $f_{m,b}$ has an inverse $f_{\frac{1}{m}, \frac{-b}{m}} \in A$. //

(11) Show that $\left\{ \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \mid m \in \mathbb{R}^\times, b \in \mathbb{R} \right\}$ is a group under matrix multiplication

Proof: Closure: $\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} n & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} mn & mc+b \\ 0 & 1 \end{bmatrix}$ has the correct form.
Associativity of matrix multiplication is well known and easy to check.
The identity is in this set: $m=1$ and $b=0$ give $\left[\begin{smallmatrix} m & b \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.
Finally closure under inverses follows from

$$\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/m & -b/m \\ 0 & 1 \end{bmatrix} \quad \text{since } 1/m \in \mathbb{R}^\times \text{ and } -b/m \in \mathbb{R}.$$

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(15) If G is a group, $g \in G$, and $g^2 = g$ then $g = e$.

Proof: If $g^2 = g$ then $g^{-1}g^2 = g^{-1}g$, so $g = g^{-1}g^2 = g^{-1}g = e$. //

(24) If a finite group G has an even number of elements there is a $g \in G$ such that $g^2 = e$. but $g \neq e$.

Proof: Note that $g^2 = e$ iff $g = g^{-1}$, so it is equivalent to look for g such that $g = g^{-1}$. Now $(g^{-1})^{-1} = g$, so G is the disjoint union of

- sets $\{x \mid x = x^{-1}\}$ and
- sets $\{x, x^{-1} \mid x \neq x^{-1}\}$.

The number of elements in sets of the second type is even, so the number of x such that $x = x^{-1}$ is even also, since the total number of elements is even. Now $e = e^{-1}$, so the

24 (cont.) number of elements x such that $x = x^{-1}$ is at least 2. Hence there is at least one such element which is not the identity. //