Coset examples [① and ② are abelian, so I list only left cosets.]

① $H = \langle 5 \rangle \triangleleft G = \mathbb{Z}_{10}$

$$\begin{aligned} H = 0 + H &= \{0,5\} = 5 + H \\ 1 + H &= \{1,6\} = 6 + H \\ 2 + H &= \{2,7\} = 7 + H \\ 3 + H &= \{3,8\} = 8 + H \\ 4 + H &= \{4,9\} = 9 + H \end{aligned}$$

② $H = \langle 10 \rangle \triangleleft G = \mathbb{Z}_{11}^{\times}$

$$\begin{aligned} H = 1H &= \{1, 10\} = 10H \\ 2H &= \{2, 9\} = 9H \\ 4H &= \{4, 7\} = 7H \\ 8H &= \{8, 3\} = 3H \\ 5H &= \{5, 6\} = 6H \end{aligned}$$

(I used powers of 2 because $\mathbb{Z}_{11}^{\times} = \langle 2 \rangle$. Any generator and its powers would also work; we could also have used the usual order $1H, 2H, 3H, 4H, \ldots$, but that has no relation to the group structure in $\mathbb{Z}_{11}^{\times}$.

NOTE: These two are exactly the same up to isomorphism! $\mathbb{Z}_{11}^{\times} \cong \mathbb{Z}_{10}$

③ $H = \langle ab \rangle < G = D_4 = \langle a, b \mid a^4 = b^2 = e,\ ba = a^3 b \rangle$

$$\begin{aligned} H = eH &= \{e, ab\} = abH \\ aH &= \{a, a^2b\} = a^2bH \\ a^2H &= \{a^2, a^3b\} = a^3bH \\ a^3H &= \{a^3, b\} = bH \end{aligned}$$

$$\begin{aligned} H = He &= \{e, ab\} = Hab \\ Ha &= \{a, b\} = Hb \\ Ha^2 &= \{a^2, a^3b\} = Ha^3b \\ Ha^3 &= \{a^3, a^2b\} = Ha^2b \end{aligned}$$

Note aH and a^3H are not right coset; Ha and Ha^3 aren't left cosets.

④ $H = \langle a^2 \rangle \triangleleft G = D_4$

$$\begin{aligned} H = eH &= \{e, a^2\} = a^2H = He = Ha^2 \\ aH &= \{a, a^3\} = a^3H = Ha = Ha^3 \\ bH &= \{b, a^2b\} = a^2bH = Hb = Ha^2b \\ abH &= \{ab, a^3b\} = a^3bH = Hab = Ha^3b \end{aligned}$$

⑤ $H = \langle (12)(34), (13)(24) \rangle \lhd S_4$

$$H = \{ (), (12)(34), (13)(24), (14)(23) \}$$
$$(12)H = \{ (12), (34), (1324), (1423) \}$$
$$(13)H = \{ (13), (1234), (24), (1432) \}$$
$$(14)H = \{ (14), (1243), (1342), (23) \}$$
$$(123)H = \{ (123), (134), (243), (142) \}$$
$$(124)H = \{ (124), (143), (132), (234) \}$$

⑥ $H = \langle (1234) \rangle < S_4$

$$H = ()H = \{ (), (1234), (13)(24), (1432) \}$$
$$(12)H = \{ (12), (234), (1324), (143) \}$$
$$(13)H = \{ (13), (12)(34), (24), (14)(23) \}$$
$$(14)H = \{ (14), (123), (1342), (243) \}$$
$$(23)H = \{ (23), (134), (1243), (142) \}$$
$$(34)H = \{ (34), (124), (1423), (132) \}$$

$$H = H() = \{ (), (1234), (13)(24), (1432) \}$$
$$H(12) = \{ (12), (134), (1423), (243) \}$$
$$H(13) = \{ (13), (14)(23), (24), (12)(34) \}$$
$$H(14) = \{ (14), (234), (1243), (132) \}$$
$$H(23) = \{ (23), (124), (1342), (143) \}$$
$$H(34) = \{ (34), (123), (1324), (142) \}$$

<u>Exercise</u>: Determine the groups

① $\mathbb{Z}_{10} / \langle 5 \rangle$

② $\mathbb{Z}_{11}^{\times} / \langle 10 \rangle$

④ $D_4 / \langle a^2 \rangle$

⑤ $S_4 / \langle (12)(34), (13)(24) \rangle$