Name: ____________________

Quiz 9
Math 2250, Fall 2015 **November 11, 2015** **R. Bruner**

1. Let $L : P_2 \longrightarrow P_2$ be $L(p(x)) = p(x) - xp'(x)$. Compute $\mathrm{Nul}(L)$ and $\mathrm{Im}(L)$. (Write them as Span of a set of polynomials. You may assume that L is linear.) (HINT: Write $p(x) = ax^2 + bx + c$ to do the calculation.)

2. Let $E : P_2 \longrightarrow \mathbf{R}$ be the error in the trapezoidal approximation to $\int_{-1}^{1} p(t)dt$:

$$E\,\cancel{L}(p(x)) = \int_{-1}^{1} p(t)dt - (p(1) + p(-1))$$

Compute $\mathrm{Nul}(\overset{E}{\cancel{L}})$. (Extra credit: Do this for P_3 rather than P_2.)

(1.) $L(ax^2+bx+c) = ax^2+bx+c - x(2ax+b)$

$= ax^2 - 2ax^2 + bx - bx + c$

$= -ax^2 + c$

$\boxed{\mathrm{Nul}(L) = \mathrm{Span}\{x\}}$ since $-ax^2+c=0 \Rightarrow a=c=0$

$\boxed{\mathrm{Im}(L) = \mathrm{Span}\{x^2, 1\}}$

(2.) $E(ax^3+bx^2+cx+d) = \int_{-1}^{1} at^3+bt^2+ct+d \; dt - (-a+b-c+d + a+b+c+d)$

$= \frac{1}{4}at^4 + \frac{1}{3}bt^3 + \frac{1}{2}ct^2 + dt \Big|_{-1}^{1} - (2b+2d)$

$= \cancel{\frac{1}{4}a - \frac{1}{4}a} + \frac{1}{3}b + \frac{1}{3}b + \cancel{\frac{1}{2}c - \frac{1}{2}c} + d + d - 2b - 2d$

$= \frac{2}{3}b - 2b + 2d - 2d = -\frac{4}{3}b$

For ax^3+bx^2+cx+d to be in Nul (E), $b=0$, So

$\mathrm{Nul}(E) = \mathrm{Span}\{1, x, x^3\}$

Another example:

Let $L: P_3 \longrightarrow P_3$ be $L(p) = p' - x p''$

i.e. $L(ax^3+bx^2+cx+d) = 3ax^2 + 2bx + c - x(6ax + 2b)$
$= -3ax^2 + c$

Then $L(p) = 0 \iff a = 0$ and $c = 0$ so

$$\mathrm{Ker}(L) = \mathrm{Nul}(L) = \mathrm{Span}\{x^2, 1\}$$

Also, any multiple of x^2 plus any multiple of 1 can be achieved as $L(p)$ for some p, so

$$\mathrm{Im}(L) = \mathrm{Span}\{x^2, 1\}.$$

In the Quiz 9 example, problem 1, it looked as if $\mathrm{Im}(L)$ was complementary to $\mathrm{Ker}(L)$: one was spanned by $\{x\}$, the other by $\{1, x^2\}$, the other basis vectors.

Here we see that was just an accident: the Image and Null Space are spanned by the same basis vectors.

Still another example

$$L : P_2 \longrightarrow P_2 \quad \text{by} \quad L(p) = p - x\,p(1)$$

i.e. $L(ax^2+bx+c) = ax^2+bx+c - (a+b+c)x$

$$= ax^2 + (-a-c)x + c$$

$$= a(x^2-x) + c(-x+1)$$

$\mathrm{Nul}(L) = \mathrm{Span}\{x\}$

$\mathrm{Im}(L) = \mathrm{Span}\{x^2-x,\ -x+1\}$

This shows that $\mathrm{Nul}(L)$ and $\mathrm{Im}(L)$ may not be spanned by basis vectors.