Name: ______________________

Worksheet and Quiz 6
Math 2250, Fall 2015 **October 21, 2015** **R. Bruner**

The matrix A has reduced row echelon form R:

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 3 & 6 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 & 2 \\ 2 & 4 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find a basis $\mathcal{C}$ for the column space Col(A).

2. Find the coordinates $[x]_{\mathcal{C}}$ for the vector $\begin{bmatrix} 4 \\ 12 \\ 5 \\ 8 \end{bmatrix}$.

3. What are the dimensions Dim(Col(A)) and Dim(Nul(A))?

Pivots are in columns 1, 3 and 4.

1. $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$, columns 1,3 and 4 of A.

2. $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 3 & 1 & 2 & 12 \\ 1 & 0 & 0 & 5 \\ 2 & 1 & 1 & 8 \end{array}\right]$ gives
$$\begin{aligned} x_1 + x_2 \quad &= 4 \\ 3x_1 + x_2 + 2x_3 &= 12 \\ x_1 \quad &= 5 \\ 2x_1 + x_2 + x_3 &= 8 \end{aligned}$$

so $x_1 = 5$, $x_2 = 4 - x_1 = -1$ and $x_3 = 8 - 2x_1 - x_2 = 8 - 10 + 1 = -1$

$$\begin{bmatrix} 4 \\ 12 \\ 5 \\ 8 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$$

i.e. $5\begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 5 \\ 8 \end{bmatrix}$

3. Dim(Col(A)) = 3 (# vectors in $\mathcal{C}$)

Dim(Nul(A)) = 5 - 3 = 2

4. Find a basis $\mathcal{N}$ for the null space Nul(A).

5. Find the coordinates $[x]_{\mathcal{N}}$ for the vector $\begin{bmatrix} -4 \\ 3 \\ -2 \\ -2 \end{bmatrix}$.

ORIGINAL WRONG VERSION

6. Find a matrix N with Col(N) = Nul(A).

From the row reduced form we get that $Ax = 0$ is the same as

$$\begin{aligned} x_1 + 2x_2 \quad\quad + 2x_5 &= 0 \\ x_3 \quad\quad - 2x_5 &= 0 \\ x_4 \quad - 2x_5 &= 0 \end{aligned}$$

or

$$\begin{aligned} x_1 &= -2x_2 - 2x_5 \\ x_2 &= x_2 \\ x_3 &= 2x_5 \\ x_4 &= 2x_5 \\ x_5 &= x_5 \end{aligned}$$

or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}$

4. $\mathcal{N} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}$

5. $x_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ -2 \\ -2 \end{bmatrix}$ is impossible: the left hand side is in $\mathbb{R}^5$ and the right hand side is in $\mathbb{R}^4$.

6. $N = \begin{bmatrix} -2 & -2 \\ 1 & 0 \\ 0 & 2 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}$

4. Find a basis $\mathcal{N}$ for the null space Nul(A).

5. Find the coordinates $[x]_{\mathcal{N}}$ for the vector $\begin{bmatrix} -4 \\ 3 \\ -2 \\ -2 \\ -1 \end{bmatrix}$.

CORRECTED VERSION

6. Find a matrix N with Col$(N) = $ Nul(A).

From R we find that $Ax=0$ if

$$\begin{aligned} x_1 + 2x_2 + 2x_5 &= 0 \\ x_3 + 2x_5 &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned}$$

OR

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ +2 \\ 2 \\ 1 \end{bmatrix}$$

4. $$\boxed{\mathcal{N} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}}$$

5. $$\left[\begin{array}{cc|c} -2 & -2 & -4 \\ 1 & 0 & 3 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \\ 0 & 1 & -1 \end{array}\right]$$ gives $x_1 = 3$, $x_2 = -1$ and some harder to interpret equations

$$\begin{bmatrix} -4 \\ 3 \\ -2 \\ -2 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\boxed{\text{Thus } \begin{bmatrix} -4 \\ 3 \\ -2 \\ -2 \\ -1 \end{bmatrix}_{\mathcal{N}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}}$$

6. $$\boxed{\begin{bmatrix} -2 & -2 \\ 1 & 0 \\ 0 & 2 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}}$$

7. (Extra Credit) Can you find a matrix C with $\mathrm{Nul}(C) = \mathrm{Col}(A)$?

Yes. We have $b = Ax$, or $b \in \mathrm{Col}(A)$ exactly when the row reduction

$$[A \mid b] \longrightarrow \left[\begin{array}{ccccc|c} & & \cdots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & * \end{array}\right] \leftarrow \text{produces } 0 \text{ in the}$$

last entry. Do this with variable coefficients

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & b_1 \\ 3 & 6 & 1 & 2 & 0 & b_2 \\ 1 & 2 & 0 & 0 & 2 & b_3 \\ 2 & 4 & 1 & 1 & 0 & b_4 \end{array}\right] \begin{array}{c} \xrightarrow{-3R_1} \\ \xrightarrow{-R_1} \\ \xrightarrow{-2R_1} \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & b_1 \\ 0 & 0 & -2 & 2 & 0 & b_2 - 3b_1 \\ 0 & 0 & -1 & 0 & 2 & b_3 - b_1 \\ 0 & 0 & -1 & 1 & 0 & b_4 - 2b_1 \end{array}\right] \xrightarrow{* -1/2}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & b_1 \\ 0 & 0 & 1 & -1 & 0 & (3b_1 - b_2)/2 \\ 0 & 0 & -1 & 0 & 2 & b_3 - b_1 \\ 0 & 0 & -1 & 1 & 0 & b_4 - 2b_1 \end{array}\right] \xrightarrow[+R_2]{} \left[\begin{array}{ccccc|c} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \hline 0 & 0 & 0 & 0 & 0 & b_4 - 2b_1 + (3b_1 - b_2)/2 \end{array}\right]$$

So the condition for b in $\mathrm{Col}(A)$ is that

$$b_4 - 2b_1 + \frac{3}{2}b_1 - \frac{b_2}{2} = 0$$

OR $\quad b_4 - \frac{1}{2}b_1 - \frac{1}{2}b_2 = 0$

OR $\quad b_1 + b_2 - 2b_4 = 0$

OR $\quad \begin{bmatrix} 1 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = 0$

$$\boxed{C = \begin{bmatrix} 1 & 1 & 0 & -2 \end{bmatrix} : \mathbb{R}^4 \longrightarrow \mathbb{R}}$$