Some additional practice
Math 2250, Fall 2015 **R. Bruner**

Here are some additional abstract vector space problems, involving vector spaces of polynomials or functions.

- Section 4.1 # 19, 21, 22.
- Section 4.2 # 31, 32, 33, 34.
- Section 4.3 # 33, 34, 35, 37, 38.
- p. 221: Example 6.
- Section 4.4 # 13, 14, 27-34.
- Section 4.5 # 21, 22, 23, 24, 34.
- Section 4.6 # 19 - 26, 30.
- Section 4.7 # 1, 13, 14, 17, 18.

Comments on the text

Here are three famous very basic differential equations whose solutions are the null spaces of linear transformations.

1. (Example 8 in Section 4.2 on p. 205) The book calls the spaces involved V and W. I would use more descriptive names:

$$D : C^1[a,b] \longrightarrow C^0[a,b]$$

sends a function f with a continuous derivative to that derivative: $D(f) = f'$. [For example, $D(x) = 1$, $D(x^2) = 2x$, etcetera. These are in $C^\infty(\mathbf{R})$. The function $f(x) = x|x|$ gives an $f \in C^1(\mathbf{R})$ since its derivative $f'(x) = |x|$ is continuous but not differentiable. That is, $D(f) = f' \in C^0(\mathbf{R})$.]

We claim that $\mathrm{Im}(D) = C^0[a,b]$ and $\mathrm{Nul}(D)$ consists of the constant functions, $\mathrm{Span}\{1\}$. The latter is familiar: a function whose derivative is 0 must be constant. The fact that D is onto, i.e. $\mathrm{Im}(D) = C^0[a,b]$, is the content of the Fundamental Theorem of Calculus: given a continuous function $f \in C^0[a,b]$, the integral (area function) gives a function

$$F(x) = \int_a^x f(t)\,dt$$

satisfying $D(F) = F' = f$. Thus $F \in C^1[a,b]$ and we have that any $f \in C^0[a,b]$ is $D(F)$ for some such F, so that D is onto.

More generally, we have

$$D : C^r[a,b] \longrightarrow C^{r-1}[a,b],$$

which sends a function f with a continuous r^{th} derivative to $D(f) = f'$, which may only be differentiable $r-1$ times. It is onto with the same null space for all r.

2. Example 9, also on p. 205, is a famous differential equation, the *simple harmonic oscillator:* the linear transformation

$$L : C^2(\mathbf{R}) \longrightarrow C^0(\mathbf{R})$$

given by

$$L(f) = f'' + \omega^2 f$$

($\omega \in \mathbf{R}$) has a two dimensional null space

$$\mathrm{Nul}(L) = \mathrm{Span}\{\cos(\omega t), \sin(\omega t)\}.$$

That is, the solutions $f(t)$ to the homogeneous equation $L(f) = f''(t) + \omega^2 f(t) = 0$ are the linear combinations

$$f(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

The equation is often written

$$f''(t) = -\omega^2 f(t)$$

to emphasize its application to Newton's Law of Motion. Note that the solutions $f(t)$ (i.e., the elements of $\mathrm{Nul}(L)$) are in $C^\infty(\mathbf{R})$, not just $C^2(\mathbf{R})$.

As with the derivative, this linear transformation L is onto: $\mathrm{Im}(L) = C^0(\mathbf{R})$.

We could use complex functions to write these solutions more simply as

$$\mathrm{Nul}(L) = \mathrm{Span}\{e^{i\omega t}, e^{-i\omega t}\}$$

3. The simpler equation

$$f' - kf = 0$$

is the homogeneous equation associated to the linear transformation

$$L : C^1(\mathbf{R}) \longrightarrow C^0(\mathbf{R})$$

by

$$L(f) = f' - kf$$

($k \in \mathbf{R}$). This has a one dimensional null space

$$\mathrm{Nul}(L) = \mathrm{Span}\{e^{kt}\}.$$

That is, the solutions $f(t)$ to the homogeneous equation $L(f) = f'(t) - kf(t) = 0$ are the linear combinations

$$f(t) = c_1 e^{kt}.$$

4. The analog of the harmonic oscillator, but with the opposite sign, is given by the linear transformation

$$L : C^2(\mathbf{R}) \longrightarrow C^0(\mathbf{R})$$

by

$$L(f) = f'' - k^2 f.$$

This too has a two dimensional null space

$$\mathrm{Nul}(L) = \mathrm{Span}\{e^{kt}, e^{-kt}\}.$$

5. Solutions to the general linear differential equation

$$f^{(n)}(t) + \cdots + a_1(t)f'(t) + a_0(t)f(t) = 0$$

are the members of the null space of the linear transformation

$$L = D^n + a_{n-1} + \cdots + a_1 D + a_0 : C^n[a,b] \longrightarrow C^0[a,b]$$

This null space is n-dimensional and has a basis consisting of mononomials times (possibly complex) exponentials determined by the factorization of the polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Worked problems

See the separate page of solutions to check your work, after you have done it.

1. Find bases for $\mathrm{Im}(L)$ and $\mathrm{Nul}(L)$ if $L : P_2 \longrightarrow P_2$ by $L(p) = p - p(1)$.
2. Find bases for $\mathrm{Im}(L)$ and $\mathrm{Nul}(L)$ if $L : P_2 \longrightarrow \mathbf{R}$ by $L(p) = p(1)$.
3. Find bases for $\mathrm{Im}(L)$ and $\mathrm{Nul}(L)$ if $L : P_2 \longrightarrow \mathbf{R}$ by $L(p) = p(1) - p(0)$.
4. Let $E : P_2 \longrightarrow \mathbf{R}$ be the error in the midpoint approximation to $\int_{-1}^{1} p(t)dt$:

$$E(p(x)) = \int_{-1}^{1} p(t)dt - 2p(0)$$

Compute $\mathrm{Nul}(E)$. (Extra credit: Do this for P_3 rather than P_2.)