Complex eigenvalues   Let λ (and $\bar{\lambda}$) be complex eigenvalues.

If $\quad Av = \lambda v$

then $\quad A\bar{v} = \bar{\lambda}\bar{v}$ So in basis $\{v, \bar{v}\}$, A is $\begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$.

In real terms $\quad \lambda = a + ib \qquad v = x + iy$

$\qquad\qquad\qquad \bar{\lambda} = a - ib \qquad \bar{v} = x - iy$

So $\quad Ax = \frac{1}{2}A(v + \bar{v}) = \frac{1}{2}\left[(a+ib)(x+iy) + (a-ib)(x-iy)\right]$

$$= \tfrac{1}{2}\left[ax + ibx + iay - by\right] + \tfrac{1}{2}\left[ax - ibx - iay - by\right] \quad = ax - by$$

while $iAy = \frac{1}{2}A(v - \bar{v}) = \frac{1}{2}\left[(a+ib)(x+iy) - (a-ib)(x-iy)\right]$

$$= \tfrac{1}{2}\left[ax + ibx + iay - by\right] - \tfrac{1}{2}\left[ax - ibx - iay - by\right] \quad = i(bx + ay)$$

So, in basis $\{x, y\}$ $\quad A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

Ex: $\quad A = \begin{bmatrix} -9 & 20 \\ -8 & 15 \end{bmatrix}$ Char poly $\lambda^2 - 6\lambda + 25$ has roots $\lambda, \bar{\lambda} = 3 \pm 4i$

$A - (3+4i)I = \begin{bmatrix} -12-4i & 20 \\ -8 & 12-4i \end{bmatrix} \xrightarrow{\text{Row reduce}} \begin{bmatrix} 1 & \frac{-3+i}{2} \\ 0 & 0 \end{bmatrix}$ $x_1 + \left(\frac{-3+i}{2}\right)x_2 = 0$

$x_1 = \left(\frac{3-i}{2}\right)x_2$

has eigenvector $v = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ so $\quad x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \; y = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Changing to basis $\{x, y\}$

$$PAP^{-1} = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}^{-1}\begin{bmatrix} -9 & 20 \\ -8 & 15 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 13 & 9 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

$$= 5\begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$$ Scaled by 5, rotated by $\tan^{-1}\left(\frac{-4}{3}\right) \simeq -53^\circ$.