

In the following questions, let  $T$  be a linear transformation  $\mathbf{R}^2 \rightarrow \mathbf{R}^3$ .

1. If  $T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}$  and  $T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ , find  $T \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ .
2. For the same transformation  $T$ , find the matrix representation of  $T$ .
3. If  $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}$  and  $T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ , find  $T \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ .
4. (Extra credit) If  $T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}$  and  $T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ , find  $T \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ .

$$1. T \begin{bmatrix} 1 \\ 5 \end{bmatrix} = T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ -2 \end{bmatrix}$$

$$2. \begin{bmatrix} 7 & 1 \\ 1 & 1 \\ 3 & -1 \end{bmatrix} \quad (3 \text{ by } 2)$$

$$3. T \begin{bmatrix} 1 \\ 5 \end{bmatrix} = T \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ -1 \end{bmatrix}$$

4. We need to write  $\begin{bmatrix} 1 \\ 5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  so we row reduce to find  $x_1$  and  $x_2$ :

$$\left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 5 \end{array} \right] \xrightarrow{-2R_1} \left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & 3 \end{array} \right] \xrightarrow{\div -\frac{1}{3}} \left[ \begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \end{array} \right] \xrightarrow{-2R_2} \left[ \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array} \right]$$

So  $\begin{bmatrix} 1 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ . Then

$$T \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 3 T \begin{bmatrix} 1 \\ 2 \end{bmatrix} - T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 21-1 \\ 3-1 \\ 9+1 \end{bmatrix} = \begin{bmatrix} 20 \\ 2 \\ 10 \end{bmatrix}$$