Name: ______________________________

Math 2250, Fall 2011, Quiz 11 **November 18, 2011** **R. Bruner**

The set $\mathcal{B} = \{x^2+1, x^2+x, (x+1)^2\}$is a basis for P_2.

1. Find the coordinates $[1]_\mathcal{B}$, $[x]_\mathcal{B}$ and $[x^2]_\mathcal{B}$.

2. Find a dependence relation between the polynomial x^2+x+1 and the elements of $\mathcal{B}$.

1. Setup:

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, shift left, "P_\mathcal{B}"] & P_2 \arrow[l, shift left, "{[\,]_\mathcal{B}}"] \arrow[r, shift right, "{[\,]_\mathcal{S}}"'] & \mathbb{R}^3 \arrow[l, shift right, "P_\mathcal{S}"'] \end{tikzcd}$$

$\mathcal{S} = \{1, x, x^2\}$ the standard basis of P_2

Want:

$$[1]_\mathcal{B} \longleftarrow 1 \longleftarrow e_1$$
$$[x]_\mathcal{B} \longleftarrow x \longleftarrow e_2$$
$$[x^2]_\mathcal{B} \longleftarrow x^2 \longleftarrow e_3$$

Given:

$$e_1 \longmapsto x^2+1 \longmapsto \begin{bmatrix}1\\0\\1\end{bmatrix} \quad \text{since } x^2+1 = 1\cdot 1 + 0\cdot x + 1\cdot x^2$$
$$e_2 \longmapsto x^2+x \longmapsto \begin{bmatrix}0\\1\\1\end{bmatrix} \quad \text{similarly}$$
$$e_3 \longmapsto (x+1)^2 \longmapsto \begin{bmatrix}1\\2\\1\end{bmatrix}$$

So the transformation we want is the inverse of $\begin{bmatrix}1&0&1\\0&1&2\\1&1&1\end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&2&0&1&0\\1&1&1&0&0&1\end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&2&0&1&0\\0&1&0&-1&0&1\end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&0&2&1&1&-1\\0&1&0&-1&0&1\end{array}\right] \xrightarrow[\times 1/2]{-\frac{1}{2}R_2}$$

$$\left[\begin{array}{ccc|ccc}1&0&0&1/2&-1/2&1/2\\0&0&1&1/2&1/2&-1/2\\0&1&0&-1&0&1\end{array}\right] \xrightarrow{\times} \left[\begin{array}{ccc|ccc}1&0&0&1/2&-1/2&1/2\\0&1&0&-1&0&1\\0&0&1&1/2&1/2&-1/2\end{array}\right]$$

$$\boxed{[1]_\mathcal{B} = \begin{bmatrix}1/2\\-1\\1/2\end{bmatrix} \quad [x]_\mathcal{B} = \begin{bmatrix}-1/2\\0\\1/2\end{bmatrix} \quad [x^2]_\mathcal{B} = \begin{bmatrix}1/2\\1\\-1/2\end{bmatrix}}$$

2. $$[x^2+x+1]_\mathcal{B} = [x^2]_\mathcal{B} + [x]_\mathcal{B} + [1]_\mathcal{B}$$
$$= \begin{bmatrix}1/2 - 1/2 + 1/2\\ 1 + 0 - 1\\ -1/2 + 1/2 + 1/2\end{bmatrix} = \begin{bmatrix}1/2\\0\\1/2\end{bmatrix}$$

So $x^2+x+1 = \frac{1}{2}(x^2+1) + \frac{1}{2}(x+1)^2$

Thus $\boxed{(x^2+x+1) - \frac{1}{2}(x^2+1) - \frac{1}{2}(x+1)^2 = 0}$