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FINAL EXAM SOLUTIONS

FALL 2011

MATH 2250

1.

5 $$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

2. (a)

5
$$\begin{aligned} 2x_1 + 3x_2 + 6x_3 &= 1 \\ 3x_1 - 5x_2 + 3x_3 &= 12 \\ -x_1 + 4x_2 + 2x_3 &= 6 \end{aligned}$$

5 (b) $$\begin{bmatrix} 2 & 3 & 6 \\ 3 & -5 & 3 \\ -1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \\ 6 \end{bmatrix}$$

3. $$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 12 \\ 2 & 1 & 7 & 16 \\ 2 & -2 & -2 & 4 \end{array}\right] \xrightarrow[-R_1]{-R_1} \left[\begin{array}{ccc|c} 2 & 0 & 4 & 12 \\ 0 & 1 & 3 & 4 \\ 0 & -2 & -6 & -8 \end{array}\right] \xrightarrow[+2R_2]{*\frac{1}{2}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

10
$$\begin{aligned} x &= 6 - 2z \\ y &= 4 - 3z \\ z &= z \end{aligned} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

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4. (a) $$\left\{\begin{aligned} x_1 + 2x_2 + 4x_5 &= 0 \\ x_3 + 2x_5 &= 0 \\ x_4 + x_5 &= 0 \end{aligned}\right\} \quad \left\{\begin{aligned} x_1 &= -2x_2 - 4x_5 \\ x_2 &= x_2 \\ x_3 &= -2x_5 \\ x_4 &= -x_5 \\ x_5 &= x_5 \end{aligned}\right\} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

6

Basis(Null(A)) = and

(b)

6 Basis(col(A)) = col's 1,3,4 = $\left\{ \begin{bmatrix} 2 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -6 \end{bmatrix} \right\}$

4 (c) Yes. Apply A, or better, its row reduced form and you get $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

4 (d) Yes. It is $-\frac{1}{6}$(col 3).

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5. (a) $(A^{-1}BC)^{-1} = C^{-1}B^{-1}A$.

5 pts

(b) ~~Yes~~. $v = A^{-1}Bv$. Oops! Not true!

No. Only if $\underline{1}$ is an eigenvalue of $A^{-1}B$.

6.

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$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & -1 & 1 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 0 & 3 & 4 & 3 \\ 1 & -1 & -1 & 1 \end{bmatrix} \xrightarrow{*\,1/3} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 4/3 & 1 \end{bmatrix} \xrightarrow{+R_2}$$

$$\begin{bmatrix} 1 & 0 & 1/3 & 2 \\ 0 & 1 & 4/3 & 1 \end{bmatrix} \quad \left\{ \begin{aligned} x &= -\tfrac{1}{3}z - 2w \\ y &= -\tfrac{4}{3}z - w \\ z &= z \\ w &= w \end{aligned} \right\} \quad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = z\begin{bmatrix} -1/3 \\ -4/3 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

7.

$$\begin{bmatrix} 2 & 4 & 4 & 2 \\ 2 & 4 & 4 & 2 \\ 1 & 2 & -1 & -2 \\ 1 & 2 & -1 & -2 \end{bmatrix} \xrightarrow[-R_3]{*\,1/2,\ -R_1} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ & & & \\ & & & \end{bmatrix} \xrightarrow{-R_1} \begin{array}{cccc} * & & * & \\ \end{array} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ & & & \\ & & & \end{bmatrix}$$

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$$\text{Basis}(\text{Span}(\text{Col's})) = \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \\ -1 \end{bmatrix} \right\}$$

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8.

$$\begin{bmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[-R_3]{-2R_3,\ -2R_3} \begin{bmatrix} 0 & 2 & 1 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{-R_1} \begin{array}{cccccc} & & * & & * & \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & +1 \end{bmatrix}$$

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Add $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

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9. $Tv = b$ always solvable $\Rightarrow$ col space has dim 2

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$\Rightarrow$ Null space has dim $4-2=2$.

10. $\dim(\mathrm{col}(T)) = 5 - 3 = 2$ so $\mathrm{col}(T)$ is not all of

5 $\mathbb{R}^3$, so not all equations $Tv = b$ are solvable.

11. (a)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 2 & 4 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-R_2]{-2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow[+R_3]{-2R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 2 & -2 \\ 0 & 0 & -1 & -2 & 0 & +1 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow[+R_2]{+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 & 0 & +1 \\ 0 & 1 & 0 & -2 & -1 & 2 \end{array}\right] \xrightarrow{*-1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right]$$

10 $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 & -1 \\ -2 & -1 & 2 \\ 2 & 0 & -1 \end{bmatrix}$$

(b) To row reduce to I, we did 1 row swap & we multiplied

5 one row by -1, for no net effect, hence $\det = 1$.

(c) $$v = \begin{bmatrix} -1 & 2 & -1 \\ -2 & -1 & 2 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 + 4 - 3 \\ -2 - 2 + 6 \\ 2 + 0 - 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

5 check $$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4-3 \\ 6-4 \\ 8-5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

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12 (a) Ind. Doesn't Span Not a basis

Summary

	I	S	B
a	Y	N	N
b	N	N	N
c	N	N	N
d	Y	Y	Y

5 each, distrib 2 2 1

20 total

(b)
$$\begin{matrix} 2 & 2 & 4 \\ 1 & -2 & -1 \\ 1 & 2 & 3 \end{matrix}$$
col 3 = sum of col's 1 and 2

Not Ind, Doesn't span, Not a basis

(c) Not Ind (too many)

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & -2 & 13 \\ 1 & 2 & -3 \end{bmatrix} \xrightarrow{*1/2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 13 \\ 1 & 2 & -3 \end{bmatrix} \underset{-R_1}{\overset{-R_1}{\longrightarrow}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 12 \\ 0 & 1 & -4 \end{bmatrix} \xrightarrow{+3R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Not Ind, Doesn't span, Not a basis

(d)
$$\begin{pmatrix} 2 & 2 & 4 \\ 1 & -2 & 1 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{*1/2} \begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \underset{-R_1}{\overset{-R_1}{\longrightarrow}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{+3R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

INd, Spans, A basis

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13. $\det \begin{bmatrix} x & y \\ y & x \end{bmatrix} = x^2 - y^2$ so it is invertible if $|x| \neq |y|$

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i.e $x^2 - y^2 \neq 0$

14. (a) $P_B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

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$$\begin{tikzcd} & \mathbb{R}^2 \arrow[dr, "{[\,]_E}"] & \\ \mathbb{R}^2 \arrow[ur, "P_B"] \arrow[rr, "P_{E \leftarrow B}"'] & & \mathbb{R}^2 \end{tikzcd}$$

10 (b)
$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 5 \end{array}\right] \underset{-R_1}{\longrightarrow} \left[\begin{array}{cc|cc} 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{-3R_2} \left[\begin{array}{cc|cc} 0 & -1 & -2 & -7 \\ 1 & 1 & 1 & 3 \end{array}\right] \underset{+R_1}{\overset{*(-1)}{\longrightarrow}} \left[\begin{array}{cc|cc} 0 & +1 & 2 & 7 \\ 1 & 0 & -1 & -4 \end{array}\right] \qquad P_{E \leftarrow B} = \begin{bmatrix} -1 & -4 \\ 2 & 7 \end{bmatrix}$$

14 (c) $\left[\begin{bmatrix}1\\2\end{bmatrix}\right]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$, so $-\begin{bmatrix}3\\4\end{bmatrix}+2\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-3+4\\-4+6\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

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and $\left[\begin{bmatrix}2\\5\end{bmatrix}\right]_{\mathcal{C}} = \begin{bmatrix}-4\\7\end{bmatrix}$ so $-4\begin{bmatrix}3\\4\end{bmatrix}+7\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-12+14\\-16+21\end{bmatrix}=\begin{bmatrix}2\\5\end{bmatrix}$

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15.

$$\begin{tikzcd} & P_2 \arrow[dr, "{[\ ]_{\mathcal{C}}}"] & \\ \mathbb{R}^3 \arrow[ur, "P_{\mathcal{B}}"] \arrow[rr, "{\begin{bmatrix}0&1&1\\0&-2&2\\1&1&1\end{bmatrix}}"'] & & \mathbb{R}^3 \end{tikzcd}$$

$e_1 \mapsto x^2 \mapsto \begin{bmatrix}0\\0\\1\end{bmatrix}$

$e_2 \mapsto (x-1)^2 \mapsto \begin{bmatrix}+1\\-2\\1\end{bmatrix}$

$e_3 \mapsto (x+1)^2 \mapsto \begin{bmatrix}1\\2\\1\end{bmatrix}$

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16.

$$\det\begin{bmatrix}14-\lambda & -6\\30 & -13-\lambda\end{bmatrix} = (14-\lambda)(-13-\lambda)+180$$
$$= (\lambda-14)(\lambda+13)+180$$
$$= \lambda^2-\lambda-182+180$$
$$= \lambda^2-\lambda-2 = (\lambda-2)(\lambda+1)$$

3 ea

$\boxed{\text{eval's } \lambda = 2, -1}$

$\lambda=2$: $\begin{bmatrix}12 & -6\\30 & -15\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ if $12x=6y$ OR $y=2x$ $\quad\begin{bmatrix}1\\2\end{bmatrix}$

2 ea

$\lambda=-1$ $\begin{bmatrix}15 & -6\\30 & -12\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ if $15x=6y$, $5x=2y$ $\quad\begin{bmatrix}2\\5\end{bmatrix}$

eigenvectors

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17.

$$C = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & & 0 \\ & 2 & \\ 0 & & 1 \end{bmatrix}$$

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Check

$$CDC^{-1} = A \iff CD = AC$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4 & & \\ & 2 & \\ & & 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 3 & -1 & 1 \\ -3 & 1 & -3 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 1 \\ -4 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 4 & 2 & 1 \\ -4 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix} \quad \text{OKAY}$$

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18. Use the standard basis $\{1, x, x^2, x^3\}$

$1 \longmapsto 0$

$x \longmapsto x$

$x^2 \longmapsto 2x^2$

$x^3 \longmapsto 3x^3$

eigenvalues $0, 1, 2, 3$ with eigenpoly's $1, x, x^2, x^3$.

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