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Math 2250, Fall 2011, Final Exam

Instructions:

1. *For full credit, explain your reasoning.*
2. Quickly read the whole test before beginning work. There are 18 questions.
3. There are 200 points, distributed as indicated.
4. The following notation will be used throughout this test:

   $\mathbf{R}$ is the real numbers.

   $\mathbf{R}^n$ is the vector space of dimension n.

   P_k is the vector space of all polynomials of degree at most k.

   Each of these has the usual addition and scalar product.

Questions:

1. (5) Find the matrix for the linear transformation $A : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$ such that

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

2. (5 pts each) Write the vector equation

$$x_1\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix} + x_3\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \\ 6 \end{bmatrix}$$

   (a) as a system of linear equations

   (b) in matrix form.

3. (10) Find all solutions to the system of linear equations

$$\begin{array}{rcrcrcr} 2x & & & + & 4z & = & 12 \\ 2x & + & y & + & 7z & = & 16 \\ 2x & - & 2y & - & 2z & = & 4 \end{array}$$

4. (20) The matrix A and its reduced row echelon form are

$$A = \begin{bmatrix} 2 & 4 & 0 & 0 & 8 \\ 3 & 6 & -3 & 0 & 6 \\ 3 & 6 & 0 & 0 & 12 \\ 3 & 6 & -3 & -6 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Give a basis for null(A).

(b) Give a basis for col(A).

(c) Is $\begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \\ -1 \end{bmatrix}$ in null(A)?

(d) Is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ in col(A)?

5. (10) Suppose that A, B and C are invertible n by n matrices.

(a) What is $(A^{-1}BC)^{-1}$?

(b) Can you solve $Av = Bv$ for every n dimensional vector v?

6. (10) Find all $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ such that $x + 2y + 3z + 4w = 0$ and $x - y - z + w = 0$.

7. (10) Find a basis for the span of

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} \right\}.$$

8. (10) Add vectors to the set

$$\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

to form a basis for $\mathbf{R}^4$.

9. (5) If $T : \mathbf{R}^4 \longrightarrow \mathbf{R}^2$ and the equation $Tv = b$ is solvable for every b in $\mathbf{R}^2$, what is $\dim(\text{null}(T))$?

10. (5) If $T : \mathbf{R}^5 \longrightarrow \mathbf{R}^3$ has a 3 dimensional null space, what is the dimension of the image (column space) of T? Is every equation $Tv = b$ solvable?

11. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

   (a) (10) Find A^{-1}.

   (b) (5) Find $\det(A)$.

   (c) (5) Solve $Av = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ without doing any more row reduction.

12. (5 each) For each of the following subsets of R^3, determine (i) whether it is linearly independent or not, (ii) whether it spans or not, and (iii) whether it is a basis or not.

   (a)

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \right\}.$$

   (b)

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \right\}$$

   (c)

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 13 \\ -3 \end{bmatrix} \right\}$$

   (d)

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\}$$

13. (5) For which values of x and y is the matrix $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$ invertible?

14. (20) The sets

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathcal{C} = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

are bases for $\mathbf{R}^2$.

(a) Write the matrix $P_{\mathcal{B}}$ which takes a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to the corresponding linear combination of the vectors in $\mathcal{B}$.

(b) Compute the basis change vector $P_{\mathcal{C} \leftarrow \mathcal{B}}$ which converts from $\mathcal{B}$ coordinates to $\mathcal{C}$ coordinates.

(c) What are the $\mathcal{C}$ coordinates of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$?

15. (10) The sets $\mathcal{S} = \{1, x, x^2\}$ and $\mathcal{B} = \{x^2, (x-1)^2, (x+1)^2\}$ are bases for P_2. Find the basis change matrix $P_{\mathcal{S} \leftarrow \mathcal{B}}$ which converts the $\mathcal{B}$ coordinates of a polynomial into its $\mathcal{S}$ (i.e. standard) coordinates.

16. (10) Find the eigenvalues and eigenvectors of

$$\begin{bmatrix} 14 & -6 \\ 30 & -13 \end{bmatrix}$$

17. (10) The matrix

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -3 & 1 & -3 \\ 1 & 1 & 3 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 4$, $\lambda_2 = 2$, and $\lambda_3 = 1$, with eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \text{and } v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

respectively. Find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

18. (10) Let $T : P_3 \longrightarrow P_3$ be the linear transformation which sends a polynomial $p(x)$ to the polynomial $xp'(x)$. Find the eigenvalues and eigenvectors of T.
(Perhaps we should call them eigenpolynomials.)

The End
and
Have a Merry Solstice