Solutions to Test 2, Math 2250, F'08

1. (a) Yes $(f+g)(3) = f(3)+g(3) = 0+0=0$
and $(cf)(3) = c(f(3)) = c \cdot 0 = 0$

(b) Yes $(f+g)(0) = f(0)+g(0) = f(3)+g(3) = (f+g)(3)$
and $(cf)(0) = c(f(0)) = c(f(3)) = (cf)(3)$.

(c) Yes $x_1+y_1 = x_3+y_3$ and $x_2+y_2 = 7x_5+7y_5 = 7(x_5+y_5)$
and $cx_1 = cx_3$ and $cx_2 = c(7x_5) = 7(cx_5)$

(d) No $x_2+y_2 = 7+x_5+7+y_5 = 14+(x_5+y_5)$, NOT $7+(x_5+y_5)$.

2. (a) No. $\mathbb{R}^3$ is 3 dimensional, and there are 6 vectors.

(b) Yes. $\begin{bmatrix} 1 & 5 \\ 2 & 4 \\ 3 & 3 \\ 4 & 2 \\ 5 & 1 \end{bmatrix} \xrightarrow[-5R_1]{-2R_1,\ -3R_1,\ -4R_1} \begin{bmatrix} 1 & 5 \\ 0 & -6 \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix}$ has two pivots, i.e. pivots in both columns.

(c) No. $\begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ has only 2 pivots.

3. (a) P_3 is 4 dimensional, so the subspace of P_3 which is 0 at $x=-1$ is 3 dimensional. We have 3 vectors, so they span iff they are independent. $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ But, they ~~are~~ dependent, so they do not span.

(b) $\mathbb{R}^5$ is 5 dimensional; the subspace where $x_1 = x_3$, $x_3 = x_5$, and $x_4 = 2x_2$ is then only 2 dimensional. These vectors are independent, so they span.

4.
$$\begin{array}{cc} & \\ \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 6 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_1} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to & \begin{array}{c} * \quad * \quad \longleftarrow \text{pivot columns} \\ \begin{bmatrix} \underline{1} & 3 & 1 & 0 \\ 0 & \underline{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-3R_2} \end{array} \end{array}$$

(a) $\text{Basis}(\text{Im}(A)) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \\ 0 \end{bmatrix} \right\}$

(b) Solving for nullspace: Reduced REF $\begin{bmatrix} 1 & 0 & -2 & -6 \\ 0 & 1 & 1 & 2 \end{bmatrix}$

$\cancel{r_1 \leftrightarrow r_3}$

$r_1 - 2r_3 - 6r_4 = 0$

$r_2 + r_3 + 2r_4 = 0$

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = r_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + r_4 \begin{bmatrix} 6 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$\text{Basis}(\text{Ker}(A)) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

5. $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} \xrightarrow{-R_2} \\ \\ \xrightarrow{+R_2} \end{matrix}$

$\begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[+R_3]{*-1} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix} \begin{matrix} \xrightarrow{-R_3} \\ \\ \xrightarrow{*1/2} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

$\begin{matrix} \xrightarrow{-R_4} \\ \xrightarrow{+R_4} \end{matrix} \begin{array}{c} \begin{matrix} * & * & * & * & \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & +1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$

(b) Basis of span $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ (first 4)

(a) Solving for nontrivial linear combinations

$r_1 + r_5 = 0$ Let $r_5 = -1$. Then $r_1 = 1, r_2 = -1, r_4 = 1$,

$r_2 - r_5 = 0$

$r_3 = 0$ So

$r_4 + r_5 = 0$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

6. Solvable for every b $\Rightarrow$ Image is all of $\mathbb{R}^2$ so $\dim(\operatorname{Im} A) = 2$. Then $\dim(\operatorname{Ker} A) = 5 - 2 = \boxed{3.}$

7. Basis for P_3 $\{1, x, x^2, x^3\}$. So starting with $\{x-1, x^2-x\}$ we get

$$\begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \end{array}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{+R_1} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{+R_2}$$

$$\begin{array}{cccccc} * & * & * & & & * \end{array}$$
$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\{x-1,\ x^2-x,\ 1,\ x^3\}}$$

8.
$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 2 & 0 \\ 3 & 2 & 0 \\ 4 & 2 & 0 \end{bmatrix}$$

9.
$$\begin{array}{cccc} x+1 & x^2+x & x^3+x^2 & x^3-3x^2-2x+2 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{-R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{-R_2}$$

vector of coefficients:
$$\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

That is,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & +1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 = 2$

$v_2 = -4$

$v_3 = 1$

$x^3 - 3x^2 - 2x + 2 =$
$2(x+1) - 4(x^2+x)$
$+ (x^3+x^2)$