R. Bruner
Math 2250, Fall 2008, Test 2
November 7, 2008

Instructions:

1. *For full credit, explain your reasoning.*
2. Quickly read the whole test before beginning work.
3. The following notation will be used throughout this test:

   $\mathbf{R}$ is the real numbers.

   $\mathbf{R}^n$ is the Euclidean space of dimension n.

   P is the vector space of all polynomials.

   P_k is the vector space of all polynomials of degree at most k.

   F is the vector space of all functions $\mathbf{R} \longrightarrow \mathbf{R}$.

Questions:

1. (5 each) Each of the following is a subset of one of the vector spaces above. Determine whether or not they are subspaces and explain why.

   (a) Functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ satisfying $f(3) = 0$.

   (b) Functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ satisfying $f(0) = f(3)$.

   (c) Vectors in $\mathbf{R}^5$ satisfying $x_1 = x_3$ and $x_2 = 7x_5$.

   (d) Vectors in $\mathbf{R}^5$ satisfying $x_1 = x_3$ and $x_2 = 7 + x_5$.

2. (5 each) Are the following sets of vectors linearly independent or not?

   (a) In $\mathbf{R}^3$,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

   (b) In $\mathbf{R}^5$,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

   (c) In P_2,

$$\left\{ x^2, \;\; x^2 - 1, \;\; x^2 - 2 \right\}$$

3. (5 each) Do the following sets of vectors span the indicated subspace or not?

   (a) Polynomials in P_3 which are 0 at $x = -1$:

$$\left\{ x + 1, \;\; x^2 + x, \;\; x^2 - 1 \right\}$$

   (b) Vectors in $\mathbf{R}^5$ satisfying $x_1 = x_3 = x_5$ and $x_4 = 2x_2$:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\}$$

– Continued on reverse –

4. (5 each) Let $A : \mathbf{R}^4 \longrightarrow \mathbf{R}^4$ be

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 6 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   (a) Find a basis for the column space $\mathrm{Im}(A)$.

   (b) Find a basis for the nullspace $\mathrm{Ker}(A)$.

5. (5 each) Consider the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

   (a) Find a nontrivial linear combination of them.

   (b) Find a basis for their span.

6. (5) If $A : \mathbf{R}^5 \longrightarrow \mathbf{R}^2$ and $Ax = b$ is solvable for every $b \in \mathbf{R}^2$, what is the dimension of the nullspace $\mathrm{Ker}(A)$?

7. (5) Expand to form a basis of P_3: $\{x - 1,\ x^2 - x\}$.

8. (5) What is the matrix of a linear transformation $\mathbf{R}^3 \longrightarrow \mathbf{R}^4$ which sends

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ to } \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ to } \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ to } \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}?$$

9. (10) Find the coordinates of $x^3 - 3x^2 - 2x + 2$ relative to the basis

$$\{x + 1,\ x^2 + x,\ x^3 + x^2\}$$

   for the space of all polynomials in P_3 which are 0 at $x = -1$.

– The End –